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A Note on the Nonstationary Binary Choice Logit Model

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Abstract

This paper derives the rate and the asymptotic distribution of the MLE of the parameter of a logit model with a nonstationary covariate when the true parameter is zero. The limit distribution of the t-statistic is also given.

Keywords: binary choice model; nonstationary covariates.

JEL Classification number: C22, C25.

1 Introduction

Suppose that the observed data $(y_t, x_t')'$ is generated by

$$\begin{aligned} y_t &= 1\{y_t^* > 0\}, \\ y_t^* &= \beta_0' x_t - \varepsilon_t, \ t = 1, ..., n. \end{aligned} \tag{1}$$

The model (1) is the standard binary choice model. When the regressor x_t and the error term ε_t are stationary, it is well known that under regularity conditions the maximum likelihood estimator(MLE) of β_0 is $\sqrt{n}-$ consistent and asymptotically normal. However, if the regressors x_t are nonstationary, the MLE of β_0 do not have these standard asymptotic properties. Recently, Park and Phillips (2000) consider the model (1) in which the regressors x_t are nonstationary, *i.e.*,

$$x_t = x_{t-1} + u_t$$

and coefficient β_0 is not zero, *i.e.*,

$$\beta_0 \neq 0.$$

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Under regularity conditions, Park and Phillips (2000) obtain dual convergence rates for the MLE. In an orthogonal direction to β_0, the MLE converges to a mixed normal random variable at a rate of $n^{3/4}$ while in all other directions, the convergence rate is $n^{1/4}$.

In this note, we derive the asymptotic properties of the MLE of β_0 when the true β_0 is zero. The model we are considering in this note is quite simple. We assume that (i) x_t is univariate, (ii) $u_t \sim iid\,(0,1)$, (iii) ε_t are iid over t with the logistic distribution $P\{\varepsilon_t < x\} = \Lambda(x) = \frac{e^x}{1+e^x}$, (iv) u_t and ε_s are independent for all t and s. These restrictive assumptions are made to make derivations of asymptotics simple and short. The assumptions in this note can be relaxed at a cost of lengthy calculations and derivations.

The main result we find in this paper is that when the true β_0 is zero and the regressor of the nonstationary binary choice is nonstationary, the MLE is n-consistent and its limit distribution is similar to so-called the unit root limit distribution. This main result will be derived in the Section 2.

2 Results

The MLE $\hat{\beta}$ maximizes the following log-likelihood function,

$$l_n(\beta) = \sum_{t=1} y_t \log \Lambda(\beta x_t) + \sum_{t=1} (1-y_t)\log(1-\Lambda(\beta x_t)).$$

As well known, the log-likelihood function $l_n(\beta)$ is smooth and strictly concave with respect to the parameter β. Thus, at the MLE $\hat{\beta}$, we have

$$\frac{\partial l_n\left(\hat{\beta}\right)}{\partial \beta} = 0.$$

Define

$$S_n(\beta) = \frac{\partial l_n(\beta)}{\partial \beta} = \sum_{t=1} x_t\left(y_t - \Lambda(\beta x_t)\right), \text{ the score function,}$$

and

$$J_n(\beta) = \frac{\partial^2 l_n(\beta)}{\partial \beta^2} = -\sum_{t=1} \dot{\Lambda}(\beta x_t)\, x_t^2, \text{ the hessian function,}$$

where

$$\dot{\Lambda}(x) = \frac{e^x}{(1+e^x)^2}.$$

Now, we find the limit distributions of the score function $S_n(\beta)$ and the hessian function $J_n(\beta)$ at the true parameter $\beta_0 = 0$. For this, let $e_t = y_t - \Lambda(0) =$

$1\{\varepsilon_t < 0\} - \frac{1}{2}$. Then, $e_t \sim iid\left(0, \frac{1}{4}\right)$. By the conventional functional central limit theorem, we have

$$\begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} e_t \\ \frac{x_{[nr]}}{\sqrt{n}} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{2} B_e(r) \\ B_u(r) \end{pmatrix},$$

where $B_e(r)$ and $B_u(r)$ are two independent Brownian motions. Using this, it is easy to find that

$$\frac{1}{n} S_n(0) = \frac{1}{n} \sum_{t=1} x_t e_t \Rightarrow \frac{1}{2} \int_0^1 B_u(r)\, dB_e(r),$$

and

$$\frac{1}{n^2} J_n(0) = -\frac{1}{n^2} \sum_{t=1} \dot{\Lambda}(0) x_t^2 \Rightarrow -\frac{1}{4} \int_0^1 B_u(r)^2\, dr. \tag{2}$$

By the mean value theorem,

$$0 = \frac{S_n\left(\hat{\beta}\right)}{n} = \frac{S_n(0)}{n} + \frac{J_n(0)}{n^2} n\hat{\beta} + \left(\frac{J_n\left(\beta^+\right) - J_n(0)}{n^2}\right) n\hat{\beta},$$

where β^+ locates between zero and $\hat{\beta}$. Suppose that for some $\delta > 0$ we have

$$\sup_{|n^{1-\delta}\beta| \le 1} \left| \frac{1}{n^{2-2\delta}} \left(J_n(\beta) - J_n(0)\right) \right| = o_p(1). \tag{3}$$

Then, by Theorem 10.1 in Wooldridge (1994), $n\hat{\beta} = O_p(1)$ and

$$\begin{aligned} n\hat{\beta} &= \frac{\frac{1}{n} S_n(0)}{-\frac{1}{n^2} J_n(0)} + o_p(1) \\ &\Rightarrow 2 \left(\int_0^1 B_u(r)^2\, dr\right)^{-1} \int_0^1 B_u(r)\, dB_e(r). \end{aligned} \tag{4}$$

To prove (3), notice by the mean value theorem that

$$\begin{aligned} &\sup_{|n^{1-\delta}\beta| \le 1} \left| \frac{1}{n^{2-2\delta}} \left(J_n(\beta) - J_n(0)\right) \right| \\ = &\sup_{|n^{1-\delta}\beta| \le 1} \left| \frac{1}{n^{2-2\delta}} \left(\sum_{t=1} x_t^2 \left(\dot{\Lambda}(\beta x_t) - \dot{\Lambda}(0) \right) \right) \right| = \sup_{|n^{1-\delta}\beta| \le 1} \left| \frac{1}{n^{2-2\delta}} \left(\sum_{t=1} x_t^3 \beta \ddot{\Lambda}\left(\beta^{++} x_t\right) \right) \right|, \end{aligned}$$

where $\ddot{\Lambda}(x) = \frac{e^x}{(1+e^x)^2} - \frac{2e^x}{(1+e^x)^3}$ and β^{++} locates between zero and β. Since since $|\beta| < \frac{1}{n^{1-\delta}}$ and $\left|\ddot{\Lambda}(x)\right| \le 3$,

$$\sup_{|n^{1-\delta}\beta| \le 1} \left| \frac{1}{n^{2-2\delta}} \left(\sum_{t=1} x_t^3 \beta \ddot{\Lambda}\left(\beta^{++} x_t\right) \right) \right| \le 3 \sup_{|n^{1-\delta}\beta| \le 1} \frac{1}{n^{3-3\delta}} \sum_{t=1} |x_t|^3.$$

Choose $0 < \delta < \frac{1}{6}$. Then, $\sup_{|n^{1-\delta}\beta|\leq 1} \frac{1}{n^{3-3\delta}} \sum_{t=1}^{n} |x_t|^3 = o_p(1)$, and in consequence,

$$\sup_{|n^{1-\delta}\beta|\leq 1} \left| \frac{1}{n^{2-2\delta}} \left(J_n(\beta) - J_n(0)\right) \right| = o_p(1),$$

as required for (3). As a summary, we have the following theorem.

Theorem 1 Under the assumptions in the previous section, if $\beta_0 = 0$, then

$$\begin{aligned} n\hat{\beta} &\Rightarrow 2\left(\int_0^1 B_u(r)^2\, dr\right)^{-1} \int_0^1 B_u(r)\, dB_e(r) \\ &\equiv MN\left(0, 4\left(\int_0^1 B_u(r)^2\, dr\right)^{-1}\right), \end{aligned}$$

where notation "$\equiv$ " signifies equivalence in distribution and MN denotes a mixed normal distribution.

Notice that in a univariate nonstationary binary choice model studied by Park and Phillips (2000), when the true parameter $\beta_0 \neq 0$, the MLE is $n^{1/4}$-consistent and its limit distribution is mixed normal. When the true parameter $\beta_0 = 0$, Theorem 1 shows that the MLE has a faster convergence rate n and its limit distribution is similar to so-called "the unit root distribution". In our note, since we assume that u_t and ε_s are independent, the limit distribution is also mixed normal.

The conventional t-statistic in this case is

$$t = \frac{\hat{\beta}}{\sqrt{-J_n\left(\hat{\beta}\right)}}.$$

By (3) and (2), we have

$$-\frac{J_n\left(\hat{\beta}\right)}{n^2} = -\frac{J_n(0)}{n^2} + o_p(1) \Rightarrow \frac{1}{4}\int_0^1 B_u(r)^2\, dr.$$

Thus,

$$t = \frac{\hat{\beta}}{\sqrt{-J_n\left(\hat{\beta}\right)}} \Rightarrow N(0,1).$$

Summarizing this, we have the following corollary.

Corollary 2 Under the assumptions in the previous section, if $\beta_0 = 0$, then

$$t = \frac{\hat{\beta}}{\sqrt{-J_n\left(\hat{\beta}\right)}} \Rightarrow N(0,1)$$

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