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Cash Reservoir**

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# OPTIMAL DIVIDEND POLICY WITH MEAN-REVERTING CASH RESERVOIR[*]

Abel Cadenillas†, Sudipto Sarkar‡, and Fernando Zapatero§

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## Abstract

Motivated by empirical evidence and economic arguments, we assume that the cash reservoir of a financial corporation follows a mean reverting process. The firm must decide the optimal dividend strategy, which consists of the optimal times and the optimal amounts to pay as dividends. We model this as a stochastic impulse control problem, and succeed in finding an analytical solution. We also find a formula for the expected time between dividend payments. A crucial and surprising economic result of our paper is that, as the dividend tax rate decreases, it is optimal for the shareholders to receive smaller but more frequent dividend payments.

**Key words:** dividends, first passage time, Ornstein-Uhlenbeck process, stochastic impulse control, taxes.

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1

# 1   Introduction

The seminal contribution to dividend research was Miller and Modigliani (1961), who claimed that dividend policy was irrelevant in perfect markets because it had no impact on firm value. However, there is plenty of evidence to support the notion that dividend policy does indeed affect firm value. That is because the Miller-Modigliani assumptions are violated in real-life financial markets. The dividend literature that followed Miller and Modigliani (1961) has therefore examined various market imperfections (taxes, asymmetric information, agency problems, etc.) in order to explain the relevance of dividend policy and to identify the optimal dividend policy. Dividend policy is part of the overall financing and investment strategies, which are usually determined jointly in practice, and sub-optimal dividend policy can result in destruction of shareholder value. Moreover, the amount of money involved in dividend decisions is very large. The U.S. corporate sector paid out more than $350 billion on payouts in 1999 alone (Allen and Michaely (2001)). Since the dividend decision has to be made repeatedly, it is important to establish an optimal dividend policy.

In real life, the determination of the optimal dividend policy is the result of a trade-off between the desire to pay a dividend as high as possible (see Jensen (1986) for a justification of this objective) and the need to maintain liquidity in order to reduce the default risk (see Kim, Mauer and Sherman (1998) for an analysis of this motivation). Models that derive the optimal dividend policy have to keep a balance between these two factors. We now review some papers that address this issue.

An excellent survey of stochastic models for the optimal dividend policy can be found in Taksar (2000). Asmussen and Taksar (1997) study the optimal dividend policy of a company that tries to maximize the expected value of the total (discounted) dividend payments to be received by the shareholders. They assume that there is no fixed cost when there is a payment of dividends, and liquid assets are modeled as a Brownian motion process with drift. They solve the problem by applying the theory of singular control. Asmussen, Højgaard and Taksar (2000), Choulli, Taksar and Zhou (2001, 2003), Højgaard and Taksar (1999, 2001), Radner and Shepp (1996), and Taksar and Zhou (1998) generalize that problem in different directions by allowing a control to affect both the potential profits and the risks of the financial corporation. They apply the theory of classical-singular stochastic control (see, for instance, Cadenillas and Haussmann (1994) for theory that covers the mixed classical-singular stochastic control problem). Some of those papers consider, in addition, the case in which the rate of the dividend payments must be bounded, and then solve that problem applying the theory of classical stochastic control. As in the previous papers, Jeanblanc-Picqué and Shiryaev (1995) study the problem of a company that tries to maximize the expected total (discounted) amount of dividend payments to be received by shareholders. Their most interesting contribution is that they model, for the first time in the literature, the dividend strategy as a stochastic impulse control problem: the company faces a fixed cost each time a dividend is paid and, as a consequence, it has to choose optimally the timing and size of the payments (see, for instance, Bensoussan and Lions (1982) or Korn (1997, 1999) for the theory and some applications of stochastic impulse controls). As in most papers related to stochastic impulse control that obtain analytical solutions, Jeanblanc-Picqué and Shiryaev (1995) assume that, when there are no interventions, the liquid assets follow a

Brownian motion process with drift.

In those papers, the un-controlled cash balance follows a Brownian motion process with drift independent of the level of cash. This is a restrictive assumption - changes in the cash balance are likely to be affected by the current balance; it is very unlikely that a firm with a large cash balance will have the same cash flow (in expectation) as a firm with a small cash balance. We therefore use a more general model for the cash balance process. Our model can accommodate mean reversion, which is suggested by Jensen's (1986) Free Cash Flow Hypothesis: a high level of liquidity (a large cash balance) induces managerial inefficiencies, hence the firm is more likely to make losses, which will tend to reduce the cash balance. On the other hand, a low cash balance increases managerial efficiency, which will increase the likelihood of profits or positive cash flows, hence the cash balance is likely to rise. Thus, the cash balance is more likely to rise (fall) when its level is low (high) - this creates mean reversion in the cash balance. This kind of behavior can be captured by our cash balance process, as discussed in Section 2. Also as discussed in Section 2, our model can accommodate other cases, e.g., when the drift is proportional to the level or independent of the level (the existing models mentioned above). We find a solution to this problem and perform a financial analysis. Furthermore, we compute the optimal expected time until the next dividend payment, provided that the firm does not default before. This is important in order to compare the predictions of our model with real data.

The payment of the dividend generates two costs: a fixed cost and a proportional cost. The fixed cost represents the resources the firm has to devote to the distribution of the dividend, for an amount that is independent of the size of the dividend. The proportional cost has the obvious and very important interpretation of the tax on dividend that the shareholder has to pay. During the last months there have been hundreds of newspaper articles published on this topic (see, for instance, the article "Clear Winners: Dividend-Paying Stocks," *Wall Street Journal*, May 23, 2003). Our paper introduces a factor that has been ignored in the debate. Our most important economic result is to show that, when the dividend tax decreases, it is optimal to reduce the size of each dividend payment to be received by the shareholders (although in an annual expected basis the dividend payout increases), which produces a higher average cash-balance and a lower probability of default. This is, to our knowledge, the first time this argument in favor of the tax cut has been stated.

The paper is structured as follows. We present the model in the next section. In section 3 we characterize the value function as a solution of quasi-variational inequalities. We obtain the optimal strategy and the value function in section 4. In section 5 we present the optimal expected time until the next intervention, provided that the firm does not default before. The proof of this result is given in the Appendix. In section 6 we present some numerical results and do some comparative statics analysis. The conclusions are presented in section 7.

# 2  The Model of Dividends

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)$, which is the $P$-augmentation of the filtration generated by a one-dimensional Brownian motion $W$.

Our state variable is the cash reservoir $X$ of a firm, that follows a mean reverting process

$$(2.1) \qquad X_t \;=\; x + \int_0^t (\delta(\rho - X_s) - c)\, ds + \int_0^t \sigma\, dW_s - \sum_{n=1}^\infty I_{\{\tau_n < t\}} \xi_n,$$

where $\delta > 0$, $\rho > 0$, $c \geq 0$ and $\sigma > 0$ are constants. The constant $c \geq 0$ represents the coupon paid to bondholders, $\delta$ is the speed of mean-reversion, $\rho - c/\delta$ is the long-term mean of the process, and $\sigma$ is the volatility. As discussed above, a mean reverting cash balance process would be consistent with Jensen's Free Cash Flow Hypothesis (discussed in the Introduction). By setting $\rho = 0$ this becomes a process with cash flows (changes in cash balance) linear in the cash balance, reflecting possible scale effects. However, default would arise quickly in that case. Finally, some of the recent literature has used a model where the un-controlled drift coefficient is a constant (e.g., Højgaard and Taksar (1999), Taksar (2000), Taksar and Zhou (1998)). That model can be obtained as a special case of the model we consider here by setting $\delta = 0$, although we would have to allow $c$ to take negative values. For interpretation purposes, we will only consider positive values for $\delta$ and nonnegative values of $c$. We assume that the initial cash reservoir is $x > 0$. Besides, $\tau_n$ is the time of the $n$-th dividend payment, and $\xi_n$ is the amount of the $n$-th dividend payment.

**Definition 2.1** *A stochastic impulse control is a pair*

$$(2.2) \qquad (\mathcal{T}, \xi) = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots),$$

*where $0 \leq \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ is an increasing sequence of stopping times, and $(\xi_n)$ is a sequence of random variables such that each $\xi_n : \Omega \mapsto [0, \infty)$ is $\mathcal{F}_{\tau_n}$-measurable. We note that $X_{\tau_n^+} = X_{\tau_n} - \xi_n$, where $\xi_n \geq 0$ is a dividend distribution. We denote $\tau_0 := 0$ and $\xi_0 := 0$.*

The time of bankruptcy is the stopping time defined by

$$\theta \;:=\; \theta^{(\mathcal{T}, \xi)} \;:=\; \inf\{t \geq 0 : X(t) = 0\}.$$

We shall assume that

$$(2.3) \qquad \qquad \forall t \geq \theta : \qquad X(t) = 0.$$

Thus, the cash reservoir is given by

$$(2.4) \qquad X_t \;=\; \begin{cases} x + \int_0^t (\delta(\rho - X_s) - c)\, ds + \int_0^t \sigma\, dW_s - \sum_{n=1}^\infty I_{\{\tau_n < t\}} \xi_n & \text{if } t < \theta \\ 0 & \text{if } t \geq \theta. \end{cases}$$

This implies that

$$P\{\forall t \in [0, \infty) : X(t) \in [0, \infty)\} = 1.$$

In the absence of frictions, the firm's objective should be to maximize the value of the firm, and the dividend policy would be irrelevant. However, a number of market frictions make dividend policy relevant in real life, e.g., limited access to external capital or more costly external capital (caused by flotation costs and under-pricing), signaling and asymmetric information regarding future prospects, tax effects, shareholder risk aversion (caused by non-diversification), etc. To quote Green and Hollifield (2003, p. 177): "The fact that firms pay dividends despite their evident tax inefficiency strongly suggests agency problems and asymmetric information are important in practice." In any case, the dividend policy is clearly relevant in practice, since companies pay considerable attention to it. In the presence of frictions, it is not so obvious what the objective function should be. In the current literature on dividend policy, the usual objective seems to maximize the expected present value of the dividend stream, e.g., Kim, Mauer and Sherman (1998). Also, it seems standard in the Math Finance literature for the firm to maximize the expected present value of the dividend stream, discounted at a specified discount rate (not necessarily the risk-free interest rate), e.g., Højgaard and Taksar (1999), Taksar (2000), Taksar and Zhou (1998), etc. However, there is no provision in these models for risk aversion, which might be relevant for certain shareholders, e.g., those who are not well diversified.

We therefore choose an objective function that can accommodate risk aversion.

**Problem 2.1** *The management wants to select the pair* $(\mathcal{T}, \xi)$ *that maximizes the functional* $J$ *defined by*

$$(2.5) \qquad J(x; \mathcal{T}, \xi) := E_x \left[ \sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) I_{\{\tau_n < \theta\}} \right],$$

*where* $g : [0, \infty) \mapsto \mathbf{R}$ *is the function defined by*

$$(2.6) \qquad g(\eta) := -K + \frac{1}{\gamma} (k\eta)^{\gamma} = -K + \frac{1}{\gamma} k^{\gamma} \eta^{\gamma}.$$

*Here,* $K$ *and* $k$ *are both positive constants, and* $\gamma \in (0, 1]$.

For $\gamma < 1$ the shareholder is risk averse, and for $\gamma = 1$ the shareholder is risk neutral. The latter case is identical to maximizing the expected present value of the net dividend stream, which is the standard objective in the existing literature as discussed above. We show in Section 6 that there might be significant economic differences between these two cases ($\gamma < 1$ and $\gamma = 1$). Our utility function $g$ is more general than the one considered by Jeanblanc-Picqué and Shiryaev (1995), who assume that $\gamma = 1$. In addition, instead of the mean reverting process (2.1), Jeanblanc-Picqué and Shiryaev (1995) assume that, in the absence of intervention, the cash reservoir follows a Brownian motion with drift. As we argued above, there are strong arguments in favor of the use of a mean-reverting process to model the cash reservoir.

We shall assume that $k \in (0, 1)$, so that $1 - k$ can be interpreted as the tax rate to be paid by the shareholder (the higher the tax rate, the smaller the $k$). Since the dividend tax is paid by the shareholder, we model it as part of the utility function. The constant $K$ has the interpretation: $K = ap$, where $a$ is a suitable constant and $p$ is the fixed amount

of money that is paid whenever there is a dividend payment, independent of the size of the payment. We assume that $p$ is paid by the shareholder, so that $K$ is a fixed disutility that is incurred each time there is a dividend payment. We observe that the fixed disutility $K$ appears in the objective function and not in the dynamics of the cash reservoir. As pointed out in page 623 of Constantinides and Richard (1978), this has been a standard assumption in the cash management literature.

Finally, we point out that the setting and objective of this paper allow us to study the trade-off between the benefits to the shareholder of a higher dividend payment, and the risk arising from a higher likelihood of default generated by the higher dividend payment. As we will see in the section on numerical computations, this drives our results and brings up a new point to the debate on dividend taxation.

We observe that

$$P\left\{\forall t \in [0, \theta] : \ 0 \leq X(t) \leq Y(t)\right\} \ = \ 1,$$

where $Y$ is the stochastic process with dynamics

$$Y_t \ = \ x + \int_0^t \left(\delta(\rho - Y_s) - c\right) ds + \int_0^t \sigma \, dW_s.$$

The process $Y = \{Y(t); t \in [0, \infty)\}$ represents the cash-flow/cash reservoir of the firm in the case in which there is no payment of dividends. For each $t \in [0, \infty)$, $Y(t)$ is normally distributed with expected value

$$E[Y(t)] \ = \ e^{-\delta t}Y(0) + \left(1 - e^{-\delta t}\right)\left(\rho - \frac{c}{\delta}\right)$$

and variance

$$\text{VAR}[Y(t)] \ = \ \frac{\sigma^2}{2\delta}\left(1 - e^{-2\delta t}\right).$$

We observe that

$$E_x\left[\int_0^\infty e^{-\lambda t}X_t^2 dt\right] \ = \ E_x\left[\int_0^\theta e^{-\lambda t}X_t^2 dt\right] + E_x\left[\int_\theta^\infty e^{-\lambda t}X_t^2 dt\right]$$

(2.7)
$$\leq \ E_x\left[\int_0^\theta e^{-\lambda t}Y_t^2 dt\right] < \infty.$$

The above equation for $E[Y(t)]$ gives

$$\lim_{T \longrightarrow \infty} E_x\left[e^{-\lambda T}X(T+)\right] \ = \ \lim_{T \longrightarrow \infty} E_x\left[e^{-\lambda T}X(T+)I_{\{\theta \leq T\}}\right] + \lim_{T \longrightarrow \infty} E_x\left[e^{-\lambda T}X(T+)I_{\{\theta > T\}}\right]$$

$$\leq \ \lim_{T \longrightarrow \infty} E_x\left[e^{-\lambda T}Y(T+)I_{\{\theta > T\}}\right]$$

$$= \ \lim_{T \longrightarrow \infty} e^{-\lambda T}E_x\left[Y(T+)I_{\{\theta > T\}}\right] = 0,$$

so
(2.8)
$$\lim_{T \longrightarrow \infty} E_x\left[e^{-\lambda T}X(T+)\right] = 0.$$

6

We observe that if there were no payment of dividends, then the value of $J$ would be zero. This implies that the value of $J$ for the optimal control is nonnegative. Since we want to maximize

$$E_x \left[ \sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) I_{\{\tau_n < \theta\}} \right],$$

we should consider only those controls such that

$$(2.9) \qquad \forall T \in [0, \infty): \qquad P_x \left\{ \lim_{n \to \infty} \tau_n \leq T \wedge \theta \right\} = 0.$$

Indeed, if a control would not satisfy such condition, then either that control would not exist or the value of $J$ for such control would be $-\infty$, and therefore it cannot be optimal.

**Definition 2.2 (Admissible controls)** *We shall say that a stochastic impulse control is admissible if the condition (2.9) is satisfied. We shall denote by $\mathcal{A}(x)$ the class of admissible stochastic impulse controls.*

# 3 The Value Function

The objective is to maximize the utility resulting from the dividend. The value function

$$V : [0, \infty) \longmapsto \mathbf{R}$$

is defined by

$$(3.1) V(x) := \sup \left\{ J(x; \mathcal{T}, \xi); \ (\mathcal{T}, \xi) \in \mathcal{A}(x) \right\} = \sup_{(\mathcal{T}, \xi) \in \mathcal{A}(x)} E_x \left[ \sum_{n=1}^{\infty} e^{-\lambda \tau_n} g(\xi_n) I_{\{\tau_n < \theta\}} \right].$$

For a function $\phi : [0, \infty) \longmapsto \mathbf{R}$, we define the *maximum utility operator $M$* by

$$(3.2) \qquad M\phi(x) := \sup \left\{ \phi(x - \eta) + g(\eta): \ \eta \in [0, \infty), x - \eta \in [0, \infty) \right\}.$$

$MV(x)$ represents the value of the strategy that consists in starting with the best dividend payment, and then selecting optimally the times and the amounts of the future dividend payments. Let us consider the differential operator $\mathcal{L}$ defined by

$$(3.3) \qquad \mathcal{L}\psi(x) := \frac{1}{2}\sigma^2 \frac{d^2\psi(x)}{dx^2} + \left( \delta(\rho - x) - c \right) \frac{d\psi(x)}{dx} - \lambda \psi(x).$$

Now we intend to find the value function and an associated optimal strategy.

Suppose there exists an optimal strategy for each initial point. Then, if the process starts at $x$ and follows the optimal strategy, the expected utility associated with this optimal strategy is $V(x)$. On the other hand, if the process starts at $x$, makes immediately the best dividend payment, and then follows an optimal strategy, then the expected utility associated with this second strategy is $MV(x)$. Since the first strategy is optimal, its associated expected utility is greater or equal than the expected utility associated with the

second strategy. Furthermore, when these two expected utilities are equal, it is optimal to intervene. Hence, $V(x) \geq MV(x)$, with equality when it is optimal to intervene. In the continuation region, that is, when there are not interventions, we must have $\mathcal{L}V(x) = 0$ (this is an heuristic application of the dynamic programming principle to the problem we are considering). These intuitive observations can be applied to give a characterization of the value function. We formalize this intuition in the next two definitions and theorem.

**Definition 3.1 (QVI)** *We say that a function* $v : [0, \infty) \mapsto [0, \infty)$ *satisfies the quasi-variational inequalities for Problem 2.1 if for every* $x \in [0, \infty)$:

$$\mathcal{L}v(x) \leq 0, \tag{3.4}$$

$$v(x) \geq Mv(x), \tag{3.5}$$

$$\Big(v(x) - Mv(x)\Big)\Big(\mathcal{L}v(x)\Big) = 0, \tag{3.6}$$

$$v(0) = 0. \tag{3.7}$$

The existence and solution to quasi-variational inequalities has been studied, for instance, in Baccarin (2004), Bensoussan and Lions (1982), and Perthame (1984a, 1984b), but the theory developed in those references cannot be applied directly to the above QVI. In any case, the objective of this paper is to find analytical solutions.

We observe that a solution $v$ of the QVI separates the interval $(0, \infty)$ into two disjoint regions: a continuation region

$$\mathcal{C} := \{x \in (0, \infty) : v(x) > Mv(x) \text{ and } \mathcal{L}v(x) = 0\}$$

and an intervention region

$$\Sigma := \{x \in (0, \infty) : v(x) = Mv(x) \text{ and } \mathcal{L}v(x) < 0\}.$$

From a solution to the QVI it is possible to construct the following stochastic impulse control.

**Definition 3.2** *Let $v$ be a solution of the QVI. The following stochastic impulse control*

$$(\mathcal{T}^v, \xi^v) = (\tau_1^v, \tau_2^v, \ldots, \tau_n^v, \ldots; \xi_1^v, \xi_2^v, \ldots, \xi_n^v, \ldots)$$

*is called the QVI-control associated with $v$ (if it exists):*

$$\tau_1^v := \inf\{t \geq 0 : v(X^v(t)) = Mv(X^v(t))\} \tag{3.8}$$

$$\xi_1^v := \arg\sup\Big\{v(X^v(\tau_1^v) - \eta) + g(\eta) : \eta \in [0, \infty), X^v(\tau_1^v) - \eta \in [0, \infty)\Big\} \tag{3.9}$$

*and, for every $n \geq 2$:*

$$\tau_n^v := \inf\{t > \tau_{n-1} : v(X^v(t)) = Mv(X^v(t))\} \tag{3.10}$$

$$\xi_n^v := \arg\sup\Big\{v(X^v(\tau_n^v) - \eta) + g(\eta) : \eta \in [0, \infty), X^v(\tau_n^v) - \eta \in [0, \infty)\Big\}. \tag{3.11}$$

*Besides, we denote $\tau_0^v := 0$ and $\xi_0^v := 0$.*

This means that the management intervenes whenever $v$ and $Mv$ coincide and the size of the dividend payment is the solution of the optimization problem corresponding to $Mv(x)$.

Jeanblanc-Picqué and Shiryaev (1995) applied, for the first time in the literature, the theory of stochastic impulse control to the problem of selecting the optimal dividend strategy of a financial corporation. That theory has been applied during recent years to model and solve many problems in finance and economics. Some recent papers about the theory and applications of stochastic impulse control include Baccarin (2002, 2004), Buckley and Korn (1998), Cadenillas and Zapatero (1999, 2000), Korn (1997, 1999), Liu (2004), Øksendal and Sulem (2002), and Pliska and Suzuki (2004). Those papers differ from our paper mainly in three assumptions. First, they allow the interventions to increase or decrease the trajectory of the system, while we only allow to decrease it. In other words, in those papers $\xi_n$ is allowed to take both nonpositive and nonnegative values, while in our paper we only allow $\xi_n$ to be nonnegative. Second, they consider a horizon $[0, \infty)$ while we consider a horizon $[0, \theta]$, where $\theta$ is a stopping time determined by the stochastic impulse control. Third, the cost or utility functions of those papers are different from ours. Thus, the theory developed in those papers cannot be applied directly to our problem. We have developed the following version of the verification theorem for stochastic impulse controls.

**Theorem 3.1** *Let $v \in C^1([0, \infty); [0, \infty))$ be a solution of the QVI and let $U \in (0, \infty)$ be such that $v \in C^2([0, \infty) - \{U\}; [0, \infty))$. Suppose that for every $x \geq U$:*

$$(3.12) \qquad v(x) = \mu + \nu(x - d)^\gamma,$$

*where $\mu \in (-\infty, \infty)$, $\nu \in (0, \infty)$, and $d \in (0, U)$. Then, for every $x \in [0, \infty)$:*

$$(3.13) \qquad V(x) \leq v(x).$$

*Furthermore, if the QVI-control $(\mathcal{T}^v, \xi^v)$ corresponding to $v$ is admissible, then it is an optimal stochastic impulse control, and for every $x \in [0, \infty)$:*

$$(3.14) \qquad V(x) = v(x) = J(x; \mathcal{T}^v, \xi^v).$$

**Proof.** We note that the differentiability of $v$ implies its continuity, and therefore its boundedness in the compact interval $[0, U]$. Furthermore, $v'$ is bounded in $[0, \infty)$, because it is continuous in $[0, U]$ and for every $x \in [U, \infty): v'(x) \in [0, \nu\gamma(U - d)^{\gamma-1}]$. Let $(\mathcal{T}, \xi)$ be an admissible policy, and denote by $X = X^{(\mathcal{T}, \xi)}$ the trajectory determined by $(\mathcal{T}, \xi)$. Since we are assuming that the function $v$ takes only non-negative values, we note that

$$\forall x \geq U : \quad 0 \leq v(x) = \mu + \nu(x - d)^\gamma \leq \mu + \nu \max((x - d), 1).$$

This, together with (2.8), the boundedness of $v$ in the compact interval $[0, U]$, the Lebesgue dominated convergence theorem, and (3.7), imply that

$$(3.15) \qquad \lim_{T \longrightarrow \infty} E_x \left[ e^{-\lambda(T \wedge \theta)} v(X_{(T \wedge \theta)+}) \right] = 0.$$

Furthermore, the boundedness of $v'$ implies that

$$(3.16) \qquad E_x \left[ \int_0^\infty \{e^{-\lambda t} v'(X(t))\}^2 dt \right] < \infty.$$

We can write for every $t > 0$ and $n \in \mathbf{N}$:

$$e^{-\lambda(t\wedge\theta\wedge\tau_n)}v\big(X_{(t\wedge\theta\wedge\tau_n)+}\big) - v(X_0)$$

$$= \sum_{i=1}^{n}\left\{e^{-\lambda(t\wedge\theta\wedge\tau_i)}v\big(X_{t\wedge\theta\wedge\tau_i}\big) - e^{-\lambda(t\wedge\theta\wedge\tau_{i-1})}v\big(X_{(t\wedge\theta\wedge\tau_{i-1})+}\big)\right\}$$

$$+ \sum_{i=1}^{n}I_{\{\tau_i\leq t\wedge\theta\}}e^{-\lambda\tau_i}\left\{v(X_{\tau_i+}) - v(X_{\tau_i})\right\}.$$

Since $X$ is a continuous semimartingale in the stochastic interval $(\tau_{i-1}, \tau_i]$ and $v$ is twice continuously differentiable in $(0, \infty) - \{U\}$, we may apply an appropriate version of Itô's formula (see, for instance, section IV.45 of Rogers and Williams (1987)). Thus, for every $i \in \mathbf{N}$:

$$e^{-\lambda(t\wedge\theta\wedge\tau_i)}v\big(X_{t\wedge\theta\wedge\tau_i}\big) - e^{-\lambda(t\wedge\theta\wedge\tau_{i-1})}v\big(X_{(t\wedge\theta\wedge\tau_{i-1})+}\big)$$

$$= \int_{(t\wedge\theta\wedge\tau_{i-1},t\wedge\theta\wedge\tau_i]}e^{-\lambda s}\left\{v'(X_s)[\delta(\rho - X_s) - c] + \frac{1}{2}\sigma^2 v''(X_s) - \lambda v(X_s)\right\}ds$$

$$+ \int_{(t\wedge\theta\wedge\tau_{i-1},t\wedge\theta\wedge\tau_i]}e^{-\lambda s}v'(X_s)\sigma dW_s$$

$$= \int_{(t\wedge\theta\wedge\tau_{i-1},t\wedge\theta\wedge\tau_i]}e^{-\lambda s}\mathcal{L}v(X_s)ds + \int_{(t\wedge\theta\wedge\tau_{i-1},t\wedge\theta\wedge\tau_i]}e^{-\lambda s}v'(X_s)\sigma dW_s.$$

According to inequality (3.4),

$$e^{-\lambda(t\wedge\theta\wedge\tau_i)}v\big(X_{t\wedge\theta\wedge\tau_i}\big) - e^{-\lambda(t\wedge\theta\wedge\tau_{i-1})}v\big(X_{(t\wedge\theta\wedge\tau_{i-1})+}\big)$$

$$\leq \int_{(t\wedge\theta\wedge\tau_{i-1},t\wedge\theta\wedge\tau_i]}e^{-\lambda s}v'(X_s)\sigma dW_s.$$

We note that this inequality becomes an equality for the QVI-control associated with $v$ (see Definition 3.2). According to inequality (3.5), in the event $\{\tau_i \leq t \wedge \theta\}$ we have

$$e^{-\lambda\tau_i}\left\{v(X_{\tau_i+}) - v(X_{\tau_i})\right\} \leq -e^{-\lambda\tau_i}g(\xi_i).$$

This inequality becomes an equality for the QVI-control associated with $v$ (see Definition 3.2). Combining the above inequalities and taking expectations, we obtain

$$v(x) - E_x\left[e^{-\lambda(t\wedge\theta\wedge\tau_n)}v\big(X_{(t\wedge\theta\wedge\tau_n)+}\big)\right]$$

$$\geq E_x\left[\sum_{i=1}^{n}\left\{I_{\{\tau_i\leq t\wedge\theta\}}e^{-\lambda\tau_i}g(\xi_i) - \int_{t\wedge\theta\wedge\tau_{i-1}}^{t\wedge\theta\wedge\tau_i}e^{-\lambda s}v'(X_s)\sigma dW_s\right\}\right],$$

with equality for the QVI-control associated with $v$. From condition (2.9),

$$\lim_{n\longrightarrow\infty}\left\{v(x) - E_x\left[e^{-\lambda(t\wedge\theta\wedge\tau_n)}v\big(X_{(t\wedge\theta\wedge\tau_n)+}\big)\right]\right\} = v(x) - E_x\left[e^{-\lambda(t\wedge\theta)}v\big(X_{(t\wedge\theta)+}\big)\right].$$

10

According to (3.16),

$$\lim_{n \longrightarrow \infty} E_x \left[ \int_0^{t \wedge \theta \wedge \tau_n} e^{-\lambda s} v'(X_s) \sigma dW_s \right] = 0.$$

Thus,

$$v(x) - E_x \left[ e^{-\lambda(t \wedge \theta)} v(X_{(t \wedge \theta)+}) \right] \geq E_x \left[ \sum_{i=1}^{\infty} I_{\{\tau_i \leq t \wedge \theta\}} e^{-\lambda \tau_i} g(\xi_i) \right],$$

with equality for the QVI-control associated with $v$.

According to (3.15),

$$\lim_{t \longrightarrow \infty} \left\{ v(x) - E_x \left[ e^{-\lambda(t \wedge \theta)} v(X_{(t \wedge \theta)+}) \right] \right\} = v(x).$$

Furthermore, according to the monotone convergence theorem,

$$\lim_{t \longrightarrow \infty} E_x \left[ \sum_{i=1}^{\infty} I_{\{\tau_i \leq t \wedge \theta\}} e^{-\lambda \tau_i} g(\xi_i) \right] = E_x \left[ \sum_{i=1}^{\infty} I_{\{\tau_i < \theta\}} e^{-\lambda \tau_i} g(\xi_i) \right].$$

Hence,

$$v(x) \geq E_x \left[ \sum_{i=1}^{\infty} I_{\{\tau_i < \theta\}} e^{-\lambda \tau_i} g(\xi_i) \right],$$

with equality for the QVI-control associated with $v$. Therefore, for every $(\mathcal{T}, \xi) \in \mathcal{A}(x)$:

(3.17) 
$$v(x) \geq J(x; \mathcal{T}, \xi),$$

with equality for the QVI-control associated with $v$. $\square$

# 4 The Solution of the QVI

We conjecture that there exists an optimal solution $(\hat{\mathcal{T}}, \hat{\xi})$ characterized by two parameters $\beta, b$ with $0 \leq \beta < b < \infty$ such that the optimal strategy is to stay in the band $[0, b]$ and jump to $\beta$ when reaching $b$. That is, we conjecture that for every $i \in \mathbf{N}$:

(4.1) 
$$\hat{\tau}_i = \inf \{t \geq \hat{\tau}_{i-1} : X_t \notin (0, b)\}$$

and

(4.2) 
$$X_{\hat{\tau}_{i+}} = X_{\hat{\tau}_i} - \hat{\xi}_i = \beta I_{\{X_{\hat{\tau}_i} = b\}}.$$

In addition, we would expect that if $x > b$, then the optimal strategy would be to jump to $\beta$. Thus, the value function would satisfy

(4.3) 
$$\forall x \in [b, \infty) : \quad V(x) = V(\beta) - K + \frac{1}{\gamma} [k(x - \beta)]^{\gamma}.$$

If $V$ were differentiable in $\{b\}$, then from equation (4.3) we would get

(4.4) 
$$V'(b) = k^{\gamma}(b - \beta)^{\gamma - 1}.$$

11

If $V$ were differentiable in $\{\beta\}$, then

$$(4.5) \qquad V'(\beta) \;=\; k^{\gamma}(b-\beta)^{\gamma-1}.$$

In fact, the maximum of $V(y) - K + \frac{1}{\gamma}k^{\gamma}(b-y)^{\gamma}$ is attained at $y = \beta$. We also conjecture that the continuation region is the interval $(0, b)$, so

$$(4.6) \quad \forall x \in [0, b]: \qquad \mathcal{L}v(x) \;=\; \frac{1}{2}\sigma^2 \frac{d^2v(x)}{dx^2} + \left(\delta(\rho - x) - c\right)\frac{dv(x)}{dx} - \lambda v(x) \;=\; 0.$$

If we would allow the case $\delta = 0$, then the general solution of this ordinary differential equation would be very simple:

$$\bar{H}(y) \;=\; \bar{A}\exp\left\{\frac{c + \sqrt{c^2 + 2\sigma^2\lambda}}{\sigma^2}y\right\} + \bar{B}\exp\left\{\frac{c - \sqrt{c^2 + 2\sigma^2\lambda}}{\sigma^2}y\right\},$$

where $\bar{A}$ and $\bar{B}$ are constants. However, as discussed above, in this paper we consider only the case $\delta > 0$. We proceed as in Cadenillas, Elliott and Léger (2002) to solve the above ordinary differential equation when $\delta > 0$. Let us denote

$$(4.7) \qquad\qquad\qquad \alpha \;:=\; \frac{2}{\sigma^2}$$

$$(4.8) \qquad\qquad\qquad m \;:=\; \rho - \frac{c}{\delta}.$$

The parameter $m$ can be interpreted as an "adjusted" long-term mean and it is the key parameter in the determination of the optimal dividend policy. In general, we expect $m$ to be positive, otherwise the coupon would be too high, which means that the company would be unreasonably indebted, given its prospects. The general solution of the above ordinary differential equation is

$$(4.9) \qquad\qquad\qquad H(y) \;=\; Af(y) + Bh(y),$$

where $A$ and $B$ are real numbers,

$$(4.10) \qquad\qquad\qquad f(y) \;:=\; \sum_{n=0}^{\infty} a_{2n}(y - m)^{2n},$$

$$(4.11) \qquad\qquad\qquad h(y) \;:=\; \sum_{n=0}^{\infty} b_{2n+1}(y - m)^{2n+1},$$

and the coefficients of the series $f$ and $h$ are given by

$$a_{2n} \;=\; \begin{cases} 1 & \text{if } n = 0 \\ \alpha\frac{1}{(2n)!}\prod_{i=0}^{n-1}(2i\delta + \lambda) & \text{if } n \geq 1 \end{cases}$$

and

$$b_{2n+1} \;=\; \begin{cases} 1 & \text{if } n = 0 \\ \alpha\frac{1}{(2n+1)!}\prod_{i=0}^{n-1}((2i+1)\delta + \lambda) & \text{if } n \geq 1. \end{cases}$$

Then,

$$f'(y) := \sum_{n=0}^{\infty} p_{2n+1}(y-m)^{2n+1} \quad \text{and} \quad h'(y) := \sum_{n=0}^{\infty} q_{2n}(y-m)^{2n},$$

where the coefficients of these series are given by

$$p_{2n+1} = \alpha \frac{1}{(2n+1)!} \prod_{i=0}^{n} (2i\delta + \lambda), \quad \text{if} \quad n \geq 0,$$

and

$$q_{2n} = \begin{cases} 1 & \text{if } n = 0 \\ \alpha \frac{1}{(2n)!} \prod_{i=0}^{n-1} ((2i+1)\delta + \lambda) & \text{if } n \geq 1. \end{cases}$$

We note that the power series $f$ converges absolutely in any interval of the form $(m - M, m + M)$, where $M < \infty$. Similarly, the power series $h$ also converges absolutely in any bounded interval. We also observe that $f(m) = 1$, $h(m) = 0$, $f'(m) = 0$, and $h'(m) = 1$, so

$$A = H(m) \quad \text{and} \quad B = H'(m).$$

In summary, we conjecture that the solution is described by (4.1)-(4.2), and that the four unknowns $A, B, \beta, b$ are a solution of the system of four equations

$$(4.12) \qquad H(b) = H(\beta) - K + \frac{1}{\gamma} k^{\gamma} (b-\beta)^{\gamma},$$

$$(4.13) \qquad H'(b) = k^{\gamma} (b-\beta)^{\gamma-1},$$

$$(4.14) \qquad H'(\beta) = k^{\gamma} (b-\beta)^{\gamma-1},$$

$$(4.15) \qquad H(0) = 0.$$

In Figure 1 we show an example of a value function, and in Figure 2 we show its derivative. The parameter values for both figures are those in the first line of Table 1. Now, we are going to prove rigorously that the above conjecture is valid.

**Theorem 4.1** *Let $A, B, b, \beta$, with $0 \leq \beta < b < \infty$ be a solution of the system of equations (4.12)-(4.15). Let us define the function $v : [0, \infty) \mapsto [0, \infty)$ by*

$$(4.16) \qquad v(x) := \begin{cases} H(x) & \text{if } 0 \leq x \leq b \\ H(\beta) - K + \frac{1}{\gamma} k^{\gamma} (x-\beta)^{\gamma} & \text{if } x > b. \end{cases}$$

*If for every $x > b$:*

$$(4.17) \quad \frac{1}{2}\sigma^2 k^{\gamma} (\gamma-1)(x-\beta)^{\gamma-2} + \delta(m-x)k^{\gamma}(x-\beta)^{\gamma-1} - \lambda \left[ H(\beta) - K + \frac{1}{\gamma} k^{\gamma} (x-\beta)^{\gamma} \right] < 0,$$

*for every $x \in (0, \beta]$: the function $\Psi_x : [0, x] \mapsto \mathbf{R}$ defined by*

$$(4.18) \qquad \Psi_x(y) := H(y) - K + \frac{1}{\gamma} k^{\gamma} (x-y)^{\gamma} \quad \text{is increasing in } [0, x],$$

13

*for every $x \in (\beta, b)$: the function $\Phi_x : [0, x] \mapsto \mathbf{R}$ defined by*

$(4.19)\Phi_x(y) := H(y) - K + \frac{1}{\gamma}k^\gamma(x - y)^\gamma$ *is increasing in $[0, \beta]$ and decreasing in $[\beta, x]$,*

*and the function $\Lambda_\beta : [\beta, b] \mapsto \mathbf{R}$ defined by*

$(4.20)\qquad \Lambda_\beta(x) := H(x) + K - \frac{1}{\gamma}k^\gamma(x - \beta)^\gamma$ *is decreasing in $[\beta, b]$,*

*then $v$ is the value function of Problem 2.1. That is,*

$(4.21)\qquad v(x) = V(x) = \sup\{J(x; \mathcal{T}, \xi); (\mathcal{T}, \xi) \in \mathcal{A}(x)\}.$

*Furthermore, the optimal strategy is given by (4.1)-(4.2).*

**Proof.** We observe that if $v$ were a solution of the QVI then, according to Theorem 3.1, $v$ would be the value function and the optimal strategy would be given by (4.1)-(4.2). Indeed, $v$ is twice continuously differentiable in $[0, b) \cup (b, \infty)$, and once continuously differentiable in $[0, \infty)$. Furthermore, $v$ has the form (3.12) in $(b, \infty)$. In addition, the QVI-control associated with $v$ is admissible, because the trajectory $X$ generated by the QVI-control associated with $v$ behaves like a mean reverting process in each random interval $(\tau_n, \tau_{n+1})$ and satisfies $P\{\forall t \in (0, \infty) : X(t) \in [0, b]\} = 1$. Thus, the conditions (2.8)-(2.9) are satisfied, and the QVI-control associated with $v$ is admissible. Hence, it only remains to verify that $v$ is a solution of the QVI.

By construction of $H$, we have for every $0 \le x \le b$:

$$\mathcal{L}v(x) = \mathcal{L}H(x) = 0.$$

According to condition (4.17), for every $x > b$:

$$\mathcal{L}v(x) = \frac{1}{2}\sigma^2 k^\gamma(\gamma - 1)(x - \beta)^{\gamma-2} + \delta(m - x)k^\gamma(x - \beta)^{\gamma-1} - \lambda\left[H(\beta) - K + \frac{1}{\gamma}k^\gamma(x - \beta)^\gamma\right]$$
$$< 0.$$

Thus,

$$\mathcal{L}v(x)$$

is equal to zero in the interval $[0, b]$ and is negative in $(b, \infty)$, so inequality (3.4) is satisfied. To compute $Mv(x)$, we need to find the value of $y \in [0, x]$ that maximizes

$$H(y) - K + \frac{1}{\gamma}k^\gamma(x - y)^\gamma.$$

According to (4.18)-(4.19),

$$Mv(x) = \begin{cases} H(x) - K & \text{if } 0 \le x < \beta \\ H(\beta) - K + \frac{1}{\gamma}k^\gamma(x - \beta)^\gamma & \text{if } x \ge \beta. \end{cases}$$

14

We observe that
$$\forall x \in [0, \beta]: \quad v(x) - Mv(x) = K > 0.$$

In addition,

$$\forall x \in (\beta, b]: \quad v(x) - Mv(x) = H(x) - H(\beta) + K - \frac{1}{\gamma}k^\gamma(x - \beta)^\gamma, \quad \text{and} \quad v(b) - Mv(b) = 0.$$

According to (4.20), the function $v - Mv$ is decreasing in $(\beta, b]$ with $v(b) - Mv(b) = 0$, so $v - Mv$ is positive in $(\beta, b)$. Thus,

$$v(x) - Mv(x)$$

is equal to zero in the intervention region $[b, \infty)$, and is positive in the continuation region $[0, b)$. Thus, inequalities (3.5)-(3.6) are satisfied. Besides, $v(0) = H(0) = 0$, so (3.7) is also satisfied. Hence, $v$ is a solution of the QVI. This proves the theorem. $\square$

We conjecture that conditions (4.17)-(4.20) are satisfied for any solution of the system of equations (4.12)-(4.15). Indeed, those conditions have been satisfied for all the numerical examples that we have examined: it is very easy to check those conditions using, for instance, Maple or Mathematica. Nevertheless, we are unable to prove this conjecture, because there is not an explicit solution of the system (4.12)-(4.15).

**Remark 4.1** *In the case $\gamma = 1$, condition (4.17) can be replaced by the simpler condition*

$$(4.22) \qquad \forall x > b: \quad \delta(m - x)k - \lambda\left[H(\beta) - K + k(x - \beta)\right] < 0,$$

*and conditions (4.18)-(4.20) can be replaced by the simpler condition*

$$(4.23) \qquad \forall 0 < x < \beta: \quad H'(x) > k \qquad \text{and} \qquad \forall \beta < x < b: \quad H'(x) < k.$$

# 5 Time to Bankruptcy or to the Next Intervention

Let us define the stopping time

$$(5.1) \qquad \tau := \tau(0, b) := \inf\{t \in [0, \infty) : X(t) \notin (0, b)\}.$$

$\tau(0, b)$ represents the first time that either the firm gets bankrupt or reaches the level $b$. It is possible to prove that $P\{\tau(0, b) < \infty\} = 1$. The notation $\{X_{\tau(0,b)} = 0\}$ represents the event that the firm will get bankrupt first, and $\{X_{\tau(0,b)} = b\}$ represents the event that the firm will reach first the level $b$.

We define the Gamma function for every $\nu > 0$ by

$$\Gamma(\nu) := \int_0^\infty u^{\nu-1}e^{-u}du,$$

and the parabolic cylinder function by

$$
D_{-\nu}(x) := \quad \exp\left\{-\frac{x^2}{4}\right\} 2^{-\frac{\nu}{2}} \sqrt{\pi}
$$

$$
\left\{\frac{1}{\Gamma\left((\nu+1)/2\right)}\left(1+\sum_{k=1}^{\infty}\frac{\nu(\nu+2)\cdots(\nu+2k-2)}{3\cdot 5\cdots(2k-1)k!}\left(\frac{x^2}{2}\right)^k\right)\right.
$$

$$
\left.-\frac{x\sqrt{2}}{\Gamma\left(\frac{\nu}{2}\right)}\left(1+\sum_{k=1}^{\infty}\frac{(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{3\cdot 5\cdots(2k+1)k!}\left(\frac{x^2}{2}\right)^k\right)\right\}.
$$

We also define for every $\nu > 0$

$$
S(\nu,x,y) := \frac{\Gamma(\nu)}{\pi}\exp\left\{\frac{x^2+y^2}{4}\right\}\left(D_{-\nu}(-x)D_{-\nu}(y)-D_{-\nu}(x)D_{-\nu}(-y)\right).
$$

In addition,

$$
\mathrm{Erfi}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{v^2}\,dv = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{x^{2k+1}}{k!(2k+1)}
$$

and

$$
\mathrm{Erfid}(x,y) := \lim_{\nu\to 0}S(\nu,x,y) = \mathrm{Erfi}\left(\frac{x}{\sqrt{2}}\right)-\mathrm{Erfi}\left(\frac{y}{\sqrt{2}}\right).
$$

**Proposition 5.1** *If $0 \le x \le b < \infty$, then*

$$
(5.2) \qquad P_x\{X_{\tau(0,b)} = 0\} \;=\; \frac{\mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma},\frac{(x-m)\sqrt{2\delta}}{\sigma}\right)}{\mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma},\frac{-m\sqrt{2\delta}}{\sigma}\right)}
$$

$$
(5.3) \qquad P_x\{X_{\tau(0,b)} = b\} \;=\; \frac{\mathrm{Erfid}\left(\frac{(x-m)\sqrt{2\delta}}{\sigma},\frac{-m\sqrt{2\delta}}{\sigma}\right)}{\mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma},\frac{-m\sqrt{2\delta}}{\sigma}\right)}.
$$

*Besides, for every $\alpha > 0$:*

$$
(5.4) \qquad E_x\left[\exp\left\{-\alpha\tau(0,b)\right\}I_{\{X_{\tau(0,b)}=b\}}\right] \;=\; \frac{S\left(\frac{\alpha}{\delta},\frac{(x-m)\sqrt{2\delta}}{\sigma},\frac{-m\sqrt{2\delta}}{\sigma}\right)}{S\left(\frac{\alpha}{\delta},\frac{(b-m)\sqrt{2\delta}}{\sigma},\frac{-m\sqrt{2\delta}}{\sigma}\right)}.
$$

**Proof.** This is a consequence of changing the location and scale of the results presented in section 7.3 of Borodin and Salminen (1996). $\Box$.

Equation (5.2) gives the probability that, when the cash reservoir is $x$, the firm will reach bankruptcy before the next dividend payment. Similarly, equation (5.3) gives the probability that, when the cash reservoir is $x$, there will be a dividend payment before the firm gets bankrupt. Equation (5.4) is the Laplace transform of the time to the next dividend payment, when the company does not bankrupt before.

16

Let us denote

$$P := \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \left(\frac{\sqrt{\delta}}{\sigma}\right)^{2i+1} \frac{(-m)^{2i+1}}{i!(2i+1)} = -\mathrm{Erfi}\left(\frac{\sqrt{\delta}}{\sigma}m\right)$$

and

$$Q := \mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma}, \frac{(-m)\sqrt{2\delta}}{\sigma}\right) = \mathrm{Erfi}\left(\frac{\sqrt{\delta}}{\sigma}(b-m)\right) + \mathrm{Erfi}\left(\frac{\sqrt{\delta}}{\sigma}m\right).$$

The following Proposition gives a formula to find the expected time to the next dividend payment, when the company does not bankrupt before.

**Proposition 5.2** *If $0 \le x \le b < \infty$, then*

$$(5.5) \quad E_x\left[\tau(0,b)I_{\{X_{\tau(0,b)}=b\}}\right] = C + D \int_{\frac{\sqrt{2\delta}}{\sigma}(-m)}^{\frac{\sqrt{2\delta}}{\sigma}(x-m)} \exp\left\{\frac{w^2}{2}\right\} dw + \sum_{n=0}^{\infty} c_n(x-m)^n.$$

*Here, the sequence $\{c_n; n \in \{0,1,2,\ldots\}\}$ must satisfy*

$$(5.6) \qquad\qquad c_2 = \frac{1}{\sigma^2}\left(-\frac{P}{Q}\right),$$

*for every $k \in \{2,3,\ldots,\}$:*

$$(5.7) \qquad c_{2k} = \left(\frac{2}{\sigma^2}\right)^k \frac{(2k-2)(2k-4)(2k-6)\cdots(2)}{(2k)!} \delta^{k-1}\left(-\frac{P}{Q}\right),$$

*and for every $k \in \{0,1,2,3,\ldots,\}$:*

$$(5.8) \quad \frac{1}{2}\sigma^2(2k+3)(2k+2)c_{2k+3} - \delta(2k+1)c_{2k+1} = -\frac{2}{\sqrt{\pi}Q}\left(\frac{\sqrt{\delta}}{\sigma}\right)^{2k+1}\frac{1}{k!(2k+1)}.$$

*The constants $C$ and $D$ can be found from the equations*

$$(5.9) \qquad\qquad C + \sum_{n=0}^{\infty} c_n(-m)^n = 0$$

*and*

$$(5.10) \qquad C + D \int_{\frac{\sqrt{2\delta}}{\sigma}(-m)}^{\frac{\sqrt{2\delta}}{\sigma}(b-m)} \exp\left\{\frac{w^2}{2}\right\} dw + \sum_{n=0}^{\infty} c_n(b-m)^n = 0.$$

**Proof.** See Theorem A.2 of the Appendix. It would not have been practical to attempt to obtain (5.5) by differentiating (5.4). □

We note that equations (5.6)-(5.7) and (5.8) indicate how to compute the even and odd terms, respectively, of the sequence $\{c_n\}$.

We are specially interested in $E_\beta\left[\tau(0,b)I_{\{X_{\tau(0,b)}=b\}}\right]$. That is the expected time between interventions, when there is not bankruptcy between those times.

# 6 Numerical Examples and Economic Analysis

In this paper we present the solution to two problems. First, the problem of finding the level of the cash reservoir at which it is optimal to pay a dividend, and the optimal size of the dividend to be paid. Second, the problem of finding the expected time until the next dividend payment, when the firm does not bankrupt before. We solve both of them analytically, and now show some numerical examples.

With respect to the solution of the first problem, it involves the numerical solution of the system of four equations (4.12)-(4.15) with four unknowns: $A, B, \beta, b$. We have written a program in C to make the computations. This program uses the Newton method (see Burden and Faires (1997)), and yields immediate results in a workstation. We need to use the function

$$H''(y) = Af''(y) + Bh''(y).$$

Here,

$$f''(y) := \sum_{n=0}^{\infty} r_{2n}(y-m)^{2n} \quad \text{and} \quad h''(y) := \sum_{n=0}^{\infty} s_{2n+1}(y-m)^{2n+1},$$

where the coefficients of these series are given by

$$r_{2n} = \alpha \frac{1}{(2n)!} \prod_{i=0}^{n} (2i\delta + \lambda) \quad \text{if } n \geq 0$$

and

$$s_{2n+1} = \alpha \frac{1}{(2n+1)!} \prod_{i=0}^{n} ((2i+1)\delta + \lambda) \quad \text{if } n \geq 0.$$

In particular, $r_0 = \alpha\lambda$ and $s_1 = \alpha(\delta + \lambda)$.

With respect to the solution of the second problem, the expected time until the next dividend payment when the firm does not bankrupt before, it involves the numerical computation of the integrals and sums of infinite terms in equations (5.5)-(5.10). We use *Mathematica* to compute the integrals. We then write a program in C to compute the value of the sums of infinite terms. Fortunately, these sums converge very quickly and we only need 100 terms to approximate those series with very high accuracy.

We present some computations for the case $\gamma = 1$ ("risk-neutral shareholder") in Table 1, and for the case $\gamma = 0.8$ ("risk-averse shareholder") in Table 2. The first seven columns include the parameter values. In columns eight and nine we present the level of the cash reservoir at which it is optimal to intervene ($\beta$), and the level to which it is optimal to bring down the cash reservoir ($b$). The size of the dividend is given by the difference $b - \beta$ in column ten. In column eleven we include the expected time until the next dividend payment, when the firm does not go bankrupt before, and for a starting value in equation (5.5) of $x = \beta$. The expected time until the next dividend payment is very important for a potential empirical test of the model: in practice, for informational reasons, companies pay dividends at fixed and regular points in time. In order to compare the predictions of our model with real data we need to have an idea of the total dividend that, on average, a

company will pay over a given period of time. For that purpose, we include the annualized dividend (that we explain below) in column twelve.

We point out that the parameter $m$, defined in equation (4.8), is key in the determination of the optimal policy and its characteristics. This parameter can be interpreted as an adjusted long-term mean (it is the long-term mean net of the coupon standardized by the speed of mean-reversion). The first important observation is that the optimal dividend policy implies that $b < m$. This is a direct consequence of the mean-reversion of the cash-reservoir process $X$. It is optimal to pay a dividend below this long-term mean, since the dynamics of the process push it upward. Of course, there is a risk involved, since the likelihood of default increases. As we can see from the comparison of Table 1 (risk-neutral shareholder) and Table 2 (risk-averse shareholder), the higher the risk-aversion of the shareholder, the higher the level at which the manager should bring down the cash reservoir ($\beta$), so that the likelihood of bankruptcy is smaller.

We observe in both tables that, as expected, the size of the dividend $(b - \beta)$ increases with the fixed disutility $K$. Surprisingly, it decreases with $k$. *Equivalently, we obtain the surprising result that the size of the dividend $(b-\beta)$ decreases as the tax rate $(1-k)$ decreases.* In the case of the risk-averse shareholder of Table 2, as $k$ increases, it is optimal to keep constant the net income of the stockholder, given by $k(b - \beta)$. We also note in both tables that the time between dividend payments $E_\beta[\tau I_{\{X_\tau = b\}}] = \mu$ decreases as $k$ increases. From the discussion up to this point, it is not clear yet what is the overall effect of the tax rate $(1-k)$ in the long run: as the tax rate decreases, the size of each dividend payment decreases, but the payments are more frequent.

In order to get a fair comparison, we can annualize the dividend stream under the assumption that it is a perpetuity, i.e., we want to find the annual dividend amount that would be equivalent in present value terms to the dividend stream of size $b - \beta$ paid at intervals of size $\mu$. It can be shown that this annualized dividend amount is equal to[1]

$$D_A = \frac{(b - \beta)\lambda}{(1 + \lambda)^\mu - 1}.$$

For the risk neutral investor ($\gamma = 1$), we get $D_A = 0.0159, 0.0156$ and $0.0153$ for $k = 0.9, 0.8$ and $0.7$ respectively; that is, the annual dividend payout increases as the tax rate is reduced. For the risk averse investor ($\gamma < 1$), we get $D_A = 0.02288, 0.02283$ and $0.02277$ for $k = 0.9, 0.8$ and $0.7$ respectively; the annual dividend payout also increases, as the tax rate is reduced, but more slowly. From these numbers, it is clear that the corporate response, to a change in the tax rate, depends on the value of risk aversion of the representative shareholder: as the tax rate is reduced, the dividend amount rises, but does it more slowly when the representative shareholder is risk averse. The reason is that higher dividends

---

[1]The quantity $\frac{k(b-\beta)}{\mu}$ is the average dividend per unit time, but it ignores time value (discounting), whereas the following takes it into account: The dividend amount is $(b - \beta)$, paid at expected intervals of size $\mu$. Assuming perpetuity, the present value of this dividend stream is $\frac{(b-\beta)}{k_\mu}$, where $k_\mu$ is the discount rate for the period $\mu$. If the annual dividend is $D_A$, the present value of the perpetual stream will be $D_A/\lambda$. For the two streams to be value-equivalent, must have $\frac{(b-\beta)}{k_\mu} = \frac{D_A}{\lambda}$. The relationship between $k_\mu$ and $\lambda$ is given by $(1 + \lambda)^\mu = 1 + k_\mu$. Substituting, we get the result.

increase the probability of bankruptcy and bankruptcy is a highly undesirable event for risk-averse shareholders. This is consistent with the available empirical evidence, since dividends have generally increased when taxes have been lowered (Chetty and Saez (2004)). However, out model incorporates an additional factor in the discussion: the risk of default that also has an effect on the determination of the optimal payout policy.

The discussion above is the main economic result of the paper and sheds light on the debate on the effects of a dividend tax. In particular, it gives a crucial argument on the advantages of a dividend tax cut, recently approved by the US Congress and the object of a heated debate over the last months.

With respect to the effect of changes in $m$ (through changes in the parameters that define it), we observe that the lower the adjusted long-term mean $m$, the more frequent and larger the optimal dividend is, since there is a higher probability that the company will default. The optimal strategy becomes more aggressive, to guarantee that the shareholders receive some dividends before default arises. For this reason, larger $\rho$, smaller $c$, and larger $\delta$ (they all imply a larger $m$) require that dividends be paid at a higher level of the cash reservoir and be smaller.

The previous results hold both for $\gamma = 1$ (Table 1) and for $\gamma = 0.8$ (Table 2), however risk aversion has a dramatic effect on the size of the dividend ($b - \beta$) and on the intervention level ($b$). The case in which the shareholder that receives dividends is risk-averse (that is, $\gamma = 0.8$) is similar to the case in which the shareholder is risk-neutral ($\gamma = 1$), but the dividend is considerably smaller and the intervention happens earlier (at a lower $b$). This result is consistent with the intuition that a more risk-averse shareholder will prefer a smaller dividend so as to not risk the default of the company, although the dividend will be paid at a lower level, that is easier to achieve. This shareholder will be happy with many small payments, although they imply higher transaction costs.

With respect to the expected time to dividend payment (when the firm does not default first), our numerical results confirm our previous conclusions. As expected, the expected time to intervention is smaller the lower the intervention level, and the smaller the size of the dividend payment, as we see in Tables 1 and 2. Consistent with our previous conclusions, we observe that the key variable that determines the time to intervention is $m$.

Up to this point, we have not mentioned stock repurchases; all the results and discussions were in terms of cash dividends, because most of the payouts are still made in this form (which might be surprising, but that is a different issue). However, our results apply equally to all cash payouts, including cash dividends and stock repurchases (for the latter, we ignore the effect of differential taxation, since repurchase will attract capital gains tax as opposed to regular income tax).

# 7 Conclusions

Our main mathematical contribution is the analytical solution, for the first time in the literature, of a stochastic impulse control problem in which the dynamics are mean reverting. In addition, we allow a reward for intervention of the form $-K + \frac{1}{\gamma}(k\xi)^\gamma$, which is different (when $\gamma \in (0, 1)$) from the rewards for intervention that are standard in the papers that

provide analytical solutions to problems of stochastic impulse control. The formula that we obtain in the Appendix for the first passage time might be also of independent interest. Our main economic contribution is the presentation of a realistic model for the optimal dividend strategy, taking into account that the cash reservoir is mean-reverting. We perform a comparative statics analysis. The main economic result is the effect of the risk of default on the optimal dividend policy: a decrease in the tax rate could result in a decrease of optimal dividend payments to be received by the shareholder (although in an annualized basis it results in an expected increase, as supported by empirical evidence). To our knowledge, this is the first time this argument has been stated on the current debate on the effect of the dividend tax.

# Appendix: Expected First Passage Time

We consider a stochastic process $Y$ that satisfies the dynamics

$$\begin{aligned} (0.1) \qquad dY_t &= \delta(m - Y_t)dt + \sigma\, dW_t \\ (0.2) \qquad Y_0 &= y, \end{aligned}$$

where $\delta > 0$, $m \in (-\infty, \infty)$, and $\sigma > 0$ are constants, and $W$ is a standard Brownian motion. For $-\infty < a < y < b < \infty$, we define the stopping time

$$(0.3) \qquad \tau(a, b) := \inf\{t \in [0, \infty) : Y(t) \notin (a, b)\}.$$

We know that (see, for instance, chapter 7.3 of Borodin and Salminen (1996))

$$(0.4) \qquad P_y\{Y_{\tau(a,b)} = a\} = \frac{\mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma}, \frac{(y-m)\sqrt{2\delta}}{\sigma}\right)}{\mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma}, \frac{(a-m)\sqrt{2\delta}}{\sigma}\right)}$$

$$(0.5) \qquad P_y\{Y_{\tau(a,b)} = b\} = \frac{\mathrm{Erfid}\left(\frac{(y-m)\sqrt{2\delta}}{\sigma}, \frac{(a-m)\sqrt{2\delta}}{\sigma}\right)}{\mathrm{Erfid}\left(\frac{(b-m)\sqrt{2\delta}}{\sigma}, \frac{(a-m)\sqrt{2\delta}}{\sigma}\right)}.$$

Let us define the functionals $f$, $g$, and $h$ by

$$\begin{aligned} f(y) &:= E_y[\tau(a, b)] \\ g(y) &:= E_y\left[\tau(a, b)I_{\{Y_{\tau(a,b)}=b\}}\right] \\ h(y) &:= P_y\{Y_{\tau(a,b)} = b\}. \end{aligned}$$

We define the differential operator $\mathcal{A}$ by

$$(0.6) \qquad \mathcal{A}\psi(y) := \frac{1}{2}\sigma^2 \frac{d^2\psi(y)}{dy^2} + \delta(m - y)\frac{d\psi(y)}{dy}.$$

**Theorem A.1.** *The functional $f$ is given by*

$$\begin{aligned} f(y) &= A + B \int_{\frac{\sqrt{2\delta}}{\sigma}(a-m)}^{\frac{\sqrt{2\delta}}{\sigma}(y-m)} \exp\left\{\frac{w^2}{2}\right\} dw \\ (0.7) \qquad & \quad -\frac{1}{\delta}\int_{\frac{\sqrt{2\delta}}{\sigma}(y-m)}^{\frac{\sqrt{2\delta}}{\sigma}(b-m)}\left[\int_w^{\frac{\sqrt{2\delta}}{\sigma}(b-m)} \exp\left\{-\frac{u^2}{2}\right\} du\right]\exp\left\{\frac{w^2}{2}\right\} dw, \end{aligned}$$

*where the constants $A$ and $B$ can be found from the equations*

$$(0.8) \qquad f(a) = 0 \qquad and \qquad f(b) = 0.$$

**Proof.** This result was obtained by Keilson and Ross (1975) in the special case in which $a = m - L$ and $b = m + L$, where $L > 0$. For our more general case, it is known (see section 5.3 of Dynkin (1965) or section 7.3 of Revuz and Yor (1999)) that $f$ must satisfy the ordinary differential equation

$$(0.9) \qquad \mathcal{A}f(y) = -1.$$

The boundary conditions are obviously given by (0.8). Then, a generalization of the method of Keilson and Ross (1975) proves the Theorem. $\square$

**Theorem A.2.** *The functional $g$ is given by*

$$(0.10) \qquad g(y) = C + D \int_{\frac{\sqrt{2\delta}}{\sigma}(a-m)}^{\frac{\sqrt{2\delta}}{\sigma}(y-m)} \exp\left\{\frac{w^2}{2}\right\} dw + \sum_{n=0}^{\infty} c_n(y-m)^n.$$

*The sequence $\{c_n; n \in \{0, 1, 2, \ldots\}\}$ must satisfy*

$$(0.11) \qquad c_2 = \frac{1}{\sigma^2}\left(-\frac{P}{Q}\right),$$

*for every $k \in \{2, 3, \ldots, \}$:*

$$(0.12) \qquad c_{2k} = \left(\frac{2}{\sigma^2}\right)^k \frac{(2k-2)(2k-4)(2k-6)\cdots(2)}{(2k)!}\delta^{k-1}\left(-\frac{P}{Q}\right),$$

*and for every $k \in \{0, 1, 2, 3, \ldots, \}$:*

$$(0.13) \quad \frac{1}{2}\sigma^2(2k+3)(2k+2)c_{2k+3} - \delta(2k+1)c_{2k+1} = -\frac{2}{\sqrt{\pi}Q}\left(\frac{\sqrt{\delta}}{\sigma}\right)^{2k+1}\frac{1}{k!(2k+1)}.$$

*Here,*

$$P := \frac{2}{\sqrt{\pi}}\sum_{i=0}^{\infty}\left(\frac{\sqrt{\delta}}{\sigma}\right)^{2i+1}\frac{(a-m)^{2i+1}}{i!(2i+1)} = -Erfi\left(\frac{\sqrt{\delta}}{\sigma}(m-a)\right)$$

*and*

$$Q := Erfid\left(\frac{(b-m)\sqrt{2\delta}}{\sigma}, \frac{(a-m)\sqrt{2\delta}}{\sigma}\right) = Erfi\left(\frac{\sqrt{\delta}}{\sigma}(b-m)\right) + Erfi\left(\frac{\sqrt{\delta}}{\sigma}(m-a)\right).$$

*The constants $C$ and $D$ can be found from the boundary conditions*

$$(0.14) \qquad g(a) = 0 \qquad and \qquad g(b) = 0.$$

**Proof.** According to section 7.3 of Revuz and Yor (1999), $g$ must satisfy the ordinary differential equation

$$(0.15) \qquad \mathcal{A}g(y) = \frac{1}{2}\sigma^2\frac{d^2g(y)}{dy^2} + \delta(m-y)\frac{dg(y)}{dy} = -h(y).$$

The boundary conditions are obviously given by (0.14). The general solution of the homogeneous linear differential equation

$$\frac{1}{2}\sigma^2 \frac{d^2 u(y)}{dy^2} + \delta(m-y)\frac{du(y)}{dy} = 0$$

is

$$u(y) = C + D \int_{\frac{\sqrt{2\delta}}{\sigma}(a-m)}^{\frac{\sqrt{2\delta}}{\sigma}(y-m)} \exp\left\{\frac{w^2}{2}\right\} dw,$$

where $C$ and $D$ are constants. To find a particular solution of the inhomogeneous linear differential equation

$$\frac{1}{2}\sigma^2 \frac{d^2 g(y)}{dy^2} + \delta(m-y)\frac{dg(y)}{dy} = -h(y),$$

we try a power series

$$v(y) = \sum_{n=0}^{\infty} c_n (y-m)^n.$$

Replacing $v$ into the above ordinary differential equation, we see that the sequence $\{c_n; n \in \{0,1,2,\ldots\}\}$ must satisfy the conditions (0.11)-(0.13). Therefore, the general solution of (0.15) is given by

$$g(y) = u(y) + v(y).$$

The constants $C$ and $D$ can be found from (0.14). $\square$

We observe that we have the freedom to choose $c_0$ and $c_1$. In particular, we can select $c_0 = 0$ and $c_1 = 0$.

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| $\rho$ | $\lambda$ | $\sigma$ | $\delta$ | $c$ | $k$ | $K$ | $\beta$ | $b$ | $b-\beta$ | $\mu$ | $D_A$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5075 | 0.6912 | 0.1837 | 8.16 | 0.0156 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.007 | 0.4962 | 0.7026 | 0.2064 | 9.17 | 0.0148 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.003 | 0.5223 | 0.6764 | 0.1541 | 6.85 | 0.0167 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.9 | 0.005 | 0.5112 | 0.6876 | 0.1764 | 7.84 | 0.0159 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.7 | 0.005 | 0.5032 | 0.6956 | 0.1924 | 8.55 | 0.0153 |
| 2 | 0.1 | 0.25 | 0.1 | 0.11 | 0.8 | 0.005 | 0.4358 | 0.6284 | 0.1926 | 6.93 | 0.0206 |
| 2 | 0.1 | 0.25 | 0.1 | 0.09 | 0.8 | 0.005 | 0.5805 | 0.7536 | 0.1731 | 9.04 | 0.0127 |
| 2 | 0.1 | 0.25 | 0.11 | 0.1 | 0.8 | 0.005 | 0.5843 | 0.7552 | 0.1709 | 8.78 | 0.0130 |
| 2 | 0.1 | 0.25 | 0.09 | 0.1 | 0.8 | 0.005 | 0.4160 | 0.6125 | 0.1965 | 6.92 | 0.0211 |
| 2 | 0.1 | 0.3 | 0.1 | 0.1 | 0.8 | 0.005 | 0.4722 | 0.6955 | 0.2233 | 6.71 | 0.0249 |
| 2 | 0.1 | 0.2 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5509 | 0.6821 | 0.1312 | 9.62 | 0.0087 |
| 2 | 0.12 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5060 | 0.6750 | 0.1690 | 7.67 | 0.0147 |
| 2 | 0.08 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5151 | 0.7152 | 0.2001 | 8.57 | 0.0171 |
| 2.2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.6549 | 0.8149 | 0.1600 | 9.53 | 0.0108 |
| 1.8 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.3653 | 0.5655 | 0.2002 | 5.37 | 0.0299 |

**Table 1.** Given parameter values, we compute $\beta$, $b$, $b-\beta$, $\mu = E_\beta[\tau I_{\{X_\tau=b\}}]$, and $D_A$ for the case $\gamma = 1$.

| $\rho$ | $\lambda$ | $\sigma$ | $\delta$ | $c$ | $k$ | $K$ | $\beta$ | $b$ | $b-\beta$ | $\mu$ | $D_A$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5946 | 0.6040 | 0.0094 | 0.4232 | 0.02283 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.007 | 0.5922 | 0.6065 | 0.0143 | 0.6436 | 0.02261 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.003 | 0.5968 | 0.6018 | 0.0050 | 0.2237 | 0.02303 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.9 | 0.005 | 0.5951 | 0.6035 | 0.0084 | 0.3762 | 0.02288 |
| 2 | 0.1 | 0.25 | 0.1 | 0.1 | 0.7 | 0.005 | 0.5939 | 0.6047 | 0.0107 | 0.4834 | 0.02277 |
| 2 | 0.1 | 0.25 | 0.1 | 0.11 | 0.8 | 0.005 | 0.5273 | 0.5367 | 0.0094 | 0.3461 | 0.02802 |
| 2 | 0.1 | 0.25 | 0.1 | 0.095 | 0.8 | 0.005 | 0.6284 | 0.6378 | 0.0094 | 0.4595 | 0.02099 |
| 2 | 0.1 | 0.25 | 0.105 | 0.1 | 0.8 | 0.005 | 0.6314 | 0.6408 | 0.0094 | 0.4561 | 0.02114 |
| 2 | 0.1 | 0.25 | 0.09 | 0.1 | 0.8 | 0.005 | 0.5095 | 0.5189 | 0.0094 | 0.3407 | 0.02848 |
| 2 | 0.1 | 0.3 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5790 | 0.5884 | 0.0094 | 0.2912 | 0.03340 |
| 2 | 0.1 | 0.22 | 0.1 | 0.1 | 0.8 | 0.005 | 0.6048 | 0.6142 | 0.0094 | 0.5569 | 0.01723 |
| 2 | 0.12 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5858 | 0.5952 | 0.0094 | 0.4325 | 0.02245 |
| 2 | 0.08 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.6104 | 0.6198 | 0.0094 | 0.4073 | 0.02361 |
| 2.05 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.6284 | 0.6378 | 0.0094 | 0.4595 | 0.02099 |
| 1.9 | 0.1 | 0.25 | 0.1 | 0.1 | 0.8 | 0.005 | 0.5273 | 0.5367 | 0.0094 | 0.3461 | 0.02802 |

**Table 2.** Given parameter values, we compute $\beta$, $b$, $b-\beta$, $\mu = E_\beta[\tau I_{\{X_\tau=b\}}]$, and $D_A$ for the case $\gamma = 0.8$.