Appendix: Omitted Proofs of "Testing for a Unit Root in Panels with Dynamic Factors"

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Appendix: Omitted Proofs of "Testing for A Unit Root in Panels with Dynamic Factors"*

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1 Summary of Moon and Perron (2003)

This note contains the omitted proofs of Moon and Perron (2003). The notation used in this note is identical to that of Moon and Perron.

1.1 A Simple Model

The model:

$$\begin{aligned} z_{it} &= \alpha_i + z_{it}^0 \\ z_{it}^0 &= \rho_i z_{it-1}^0 + y_{it}, \end{aligned} \quad (1)$$

where $z_{i0}^0 = 0$ for all i.

Assume

$$\rho_i = 1 - \frac{\theta_i}{\sqrt{n}T}, \quad (2)$$

where θ_i is a non-negative random variable.

Assumption 1 *The random variables θ_i are iid with mean μ_θ and a finite fourth moment, and they are defined on $[0, \bar{M}_\theta]$.*

With this assumption, the hypotheses we will consider are

$$\mathbb{H}_0 : \mu_\theta = 0.$$

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against the local alternative

$$\mathbb{H}_1' : \mu_\theta > 0.$$

Under Assumption 1, the null hypothesis is equivalent to

$$\mathbb{H}_0 : \theta_i = 0 \text{ for all } i.$$

To model the correlation among the cross-sectional units, we will assume that the error term in (1) follows an approximate factor model:

$$y_{it} = \beta_i^{0\prime} f_t^0 + e_{it}, \tag{3}$$

where f_t^0 are $K-$vectors of unobservable random factors, β_i^0 are nonrandom factor loading coefficient vectors $(K - vector)$, e_{it} are idiosyncratic shocks, and the number of factors K is unknown.

Assumption 2 *(i)* $e_{it} = \sum_{j=0}^{\infty} d_{ij} v_{it-j}$, *where* v_{it} *are* $iid(0,1)$ *across* i *and over* t, *have a finite eighth moment.*

(ii) Let $\kappa_8 = E\left(v_{it}^8\right)$. *Then,* $\inf_i \sum_{j=0}^{\infty} d_{ij} > 0$.

(iii) Let $\bar{d}_j = \sup_i |d_{ij}|$ *.Then,* $\sum_{j=0}^{\infty} j^m \bar{d}_j < M$ *for some* $m > 1$.

Assumption 3 *(i)* $f_t^0 = \sum_{j=0}^{\infty} c_j u_{t-j}$, *where* c_j *are* $K \times K$ *matrices of real numbers and the* K*-vectors* u_i *are* $iid(0, I_K)$ *across* i *and over* t.

(ii) $\sum_{j=0}^{\infty} j^m \|c_j\| < M$ *for some* $m > 1$.

Assumption 4 θ_i, u_t, *and* v_{js} *are independent.*

Assumption 5 $1 \le K \le \bar{K} < \infty$, *where* $\bar{K}$ *is known.*

Assumption 6 *As* $n \to \infty$, $\frac{1}{n}\sum_{i=1}^{n} \beta_i^0 \beta_i^{0\prime} \to \Sigma_\beta > 0$.

Assumption 7 *As* $T \to \infty$, $\frac{1}{T}\sum_{t=1}^{T} f_t^0 f_t^{0\prime} \to_p \Sigma_f > 0$.

Define $\sigma_{e,i}^2 = \sum_{j=0}^{\infty} d_{ij}^2$, $\omega_{e,i}^2 = \left(\sum_{j=0}^{\infty} d_{ij}\right)^2$, and $\lambda_{e,i} = \sum_{l=1}^{\infty}\sum_{j=0}^{\infty} d_{ij} d_{ij+l}$. In this notation, $\sigma_{e,i}^2$ signifies the variance of e_{it}, $\omega_{e,i}^2$ the long-run variance of e_{it}, and $\lambda_{e,i}$ the one-sided long-run variance of e_{it}.

Assumption 8 *As* $n \to \infty$,

(i) $\omega_e^2 \overset{let}{=} \lim_n \frac{1}{n}\sum_{i=1}^{n} \omega_{e,i}^2 \, (> 0)$ *is well defined.*

(ii) $\phi_e^4 \overset{let}{=} \lim_n \frac{1}{n}\sum_{i=1}^{n} \omega_{e,i}^4 \, (> 0)$ *is well defined.*

(iii) $\sigma_e^2 \overset{let}{=} \lim_n \frac{1}{n}\sum_{i=1}^{n} \sigma_{e,i}^2 \, (> 0)$ *is well defined.*

Assumption 9 $\sup_i E\left(\alpha_i^2\right) < \infty$

We now define our matrix notation: Define

$$\begin{aligned}
y &= \left(\underline{y}_1, ..., \underline{y}_T\right), \ \underline{y}_i = (y_{i1}, ..., y_{iT})', \\
e &= \left(\underline{e}_1, ..., \underline{e}_n\right), \ \underline{e}_i = (e_{i1}, ..., e_{iT})', \\
Z &= \left(\underline{Z}_1, ..., \underline{Z}_n\right), \ \underline{Z}_i = (z_{i1}, ..., z_{iT})', \\
Z_{-1} &= \left(\underline{Z}_{-1,1}, ..., \underline{Z}_{-1,n}\right), \ \underline{Z}_{-1,i} = (z_{i0}, ..., z_{iT-1})', \\
Z^0 &= \left(\underline{Z}_1^0, ..., \underline{Z}_n^0\right), \ \underline{Z}_i^0 = \left(z_{i1}^0, ..., z_{iT}^0\right)', \\
Z_{-1}^0 &= \left(\underline{Z}_{-1,1}^0, ..., \underline{Z}_{-1,n}^0\right), \ \underline{Z}_{-1,i}^0 = \left(z_{i0}^0, ..., z_{iT-1}^0\right)', \\
f^0 &= \left(f_1^0, ..., f_T^0\right)' \ ; \beta^0 = \left(\beta_1^0, ..., \beta_n^0\right)'.
\end{aligned}$$

Define

$$\rho(L) = diag(\rho_1 L, ..., \rho_n L),$$

where L denotes a lag operator. Write $l_T = (1, ..., 1)'$, $T \times 1$ vector of ones. Using our matrix notation, we rewrite the model as

$$\begin{aligned} Z &= l_T \alpha' + Z^0, \\ Z^0 (I_n - \rho(L)) &= f^0 \beta^{0\prime} + e. \end{aligned} \tag{4}$$

1.1.1 Pooled Estimators and Their Asymptotics

Define the pooled autoregressive estimator:

$$\hat{\rho}_{pool} = \frac{tr(Z'_{-1} Z)}{tr(Z'_{-1} Z_{-1})}.$$

Lemma 1 *Suppose that Assumptions 1 – 9 hold. Then, as* $(n, T \to \infty)$, *under the null of unit root,*

$$T(\hat{\rho}_{pool} - 1) \Rightarrow \frac{\frac{1}{2} tr\left(B_f(1) B_f(1)' \Sigma_\beta\right) + \frac{1}{2}\omega_e^2 - \frac{1}{2} tr(\Sigma_f \Sigma_\beta) - \frac{1}{2}\sigma_e^2}{tr\left(\int_0^1 B_f(r) B_f(r)' dr\right) \Sigma_\beta + \frac{1}{2}\omega_e^2}.$$

Define

$$\hat{\rho}^+_{pool} = \frac{tr\left(Z_{-1} Q_{\beta^0} Z'\right) - nT\lambda_e^n}{tr\left(Z_{-1} Q_{\beta^0} Z'_{-1}\right)}, \tag{5}$$

$\lambda_e^n = \frac{1}{n}\sum_{i=1}^n \lambda_{e,i}$. Define $\Theta = diag(\theta_1, ..., \theta_n)$. Now to find the asymptotic distribution of $\hat{\rho}^+_{pool}$, we write by definition that

$$\begin{aligned} &\sqrt{n}T\left(\hat{\rho}^+_{pool} - 1\right) \\ = \ & \frac{\sqrt{n}\left(\frac{1}{nT} tr\left(Z_{-1} Q_{\beta^0} (Z - Z_{-1})'\right) - \lambda_e^n\right)}{\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} Z'_{-1}\right)} \\ = \ & \frac{\sqrt{n}\left(\frac{1}{nT} tr\left(Z_{-1} Q_{\beta^0} \left(-\frac{1}{\sqrt{n}T}\Theta Z^0_{-1} + y\right)'\right) - \lambda_e^n\right)}{\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} Z'_{-1}\right)} \\ = \ & -\frac{\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} \Theta Z^{0\prime}_{-1}\right)}{\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} Z'_{-1}\right)} + \frac{\sqrt{n}\left(\frac{1}{nT} tr\left(Z^0_{-1} Q_{\beta^0} e'\right) - \lambda_e^n\right)}{\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} Z'_{-1}\right)}. \end{aligned}$$

Lemma 2 *Suppose that Assumptions 1 – 9 hold. Assume that* $(n, T \to \infty)$ *with* $\frac{n}{T} \to 0$. *Then, the following holds.*

(a) $\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} Z'_{-1}\right) \to_p \frac{1}{2}\omega_e^2$.
(b) $\frac{1}{nT^2} tr\left(Z_{-1} Q_{\beta^0} \Theta Z^{0\prime}_{-1}\right) \to_p \frac{1}{2}\mu_\theta \omega_e^2$.
(c) $\sqrt{n}\left(\frac{1}{nT} tr\left(Z^0_{-1} Q_{\beta^0} e'\right) - \lambda_e^n\right) \Rightarrow N\left(0, \frac{1}{2}\phi_e^4\right)$.

Using the results in Lemma 2, we can derive the asymptotic distribution of $\sqrt{n}T\left(\hat{\rho}^+_{pool} - 1\right)$ as follows.

Theorem 1 *Suppose that Assumptions 1 – 9 hold. Assume that* $(n, T \to \infty)$ *with* $\frac{n}{T} \to 0$. *Then,*

$$\sqrt{n}T\left(\hat{\rho}^{+}_{pool} - 1\right) \Rightarrow N\left(-\mu_\theta, \frac{2\phi_e^4}{\omega_e^4}\right).$$

1.1.2 Panel Unit Root Test Statistics and Their Asymptotics

In view of Theorem 1 and Lemma 2 in the appendix, we may deduce that

$$\frac{\sqrt{n}T\left(\hat{\rho}^{+}_{pool} - 1\right)}{\sqrt{\frac{2\phi_e^4}{\omega_e^4}}} \Rightarrow N\left(-\mu_\theta\sqrt{\frac{\omega_e^4}{2\phi_e^4}}, 1\right) \tag{6}$$

and

$$\sqrt{n}T\left(\hat{\rho}^{+}_{pool} - 1\right)\sqrt{\frac{1}{nT^2}tr\left(Z_{-1}Q_{\beta_0}Z'_{-1}\right)\frac{\omega_e^2}{\phi_e^4}} \Rightarrow N\left(-\mu_\theta\sqrt{\frac{\omega_e^4}{2\phi_e^4}}, 1\right) \tag{7}$$

as $(n, T \to \infty)$ with $\frac{n}{T} \to 0$.

Estimation of β^0

To estimate β_i^0, we use the principal component method. In model (**??**), since the error term y_{it} is not observable, we use the residual

$$\hat{y} = Z - \hat{\rho}_{pool}Z_{-1}.$$

To estimate β^0 and f^0, we minimize

$$V_{nT}(f, \beta, K) = \frac{tr\left(\left(\hat{y} - f\beta'\right)\left(\hat{y} - f\beta'\right)'\right)}{nT}$$

with respect to $\frac{\beta'\beta}{n} = I_K$ or $\frac{f'f}{T} = I_K$. With the normalization $\frac{\beta'\beta}{n} = I_K$, we have the estimated factor loading matrix $\bar{\beta}_K$ that is a $(n \times K)$ matrix of $\sqrt{n}$ times the eigenvectors corresponding to the K largest eigenvalues of $\hat{y}'\hat{y}$. Then, we obtain an estimator of the factor, $\bar{f}_K = \frac{1}{n}\hat{y}\bar{\beta}_K$. On the other hand, if we use the normalization $\frac{f'f}{T} = I_K$, we have the estimated factor $\check{f}_K$ that is a $(T \times K)$ matrix of $\sqrt{T}$ times the eigenvectors corresponding to the K largest eigenvalues of $\hat{y}\hat{y}'$, and the estimated factor loading $\check{\beta}_K = \frac{1}{T}\hat{y}'\check{f}_K$. Define

$$\hat{\beta}_K = \check{\beta}_K\left(\frac{\check{\beta}'_K\check{\beta}_K}{n}\right)^{1/2}, \tag{8}$$

a re-scaled estimator of the factor loading[1]. This is the estimator of β^0 that we will use in defining a panel unit root test statistic.

The following lemma shows that the projection matrix $Q_{\hat{\beta}_K}$ is consistent and finds its convergence order.

[1] The rescaled estimator studied in Bai (2001) is $\bar{\beta}_K\left(\frac{\bar{f}'_K\bar{f}_K}{T}\right)^{-1/2}$.

Lemma 3 *Suppose that Assumptions 1 – 9 hold. Assume that* $(n, T \to \infty)$ *with* $\frac{n}{T} \to 0$. *Then,*

$$\left\| Q_{\hat{\beta}_K} - Q_{\beta^0} \right\| = O_p\left(\max\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right) \right).$$

Estimation of the long-run variances

In order to implement the t-statistics in (6) and (7), we also need consistent estimators, say $\hat{\lambda}_e^n$, $\hat{\omega}_e^2$, and $\hat{\phi}_e^4$, for λ_e^n, ω_e^2, and ϕ_e^4, respectively, satisfying

$$\sqrt{n}\left(\hat{\lambda}_e^n - \lambda_e^n\right) = o_p(1), \quad (9)$$

$$\hat{\omega}_e^2 - \omega_e^2 = o_p(1), \quad (10)$$

and

$$\hat{\phi}_e^4 - \phi_e^4 = o_p(1). \quad (11)$$

In this section we propose estimators of λ_e^n, ω_e^2, and ϕ_e^4 that satisfy these conditions.

Let $\hat{e}_{it}$ denote the $(t,i)^{th}$ element of $\hat{e} = \hat{y} Q_{\hat{\beta}_K}$. Define the sample covariances $\hat{\Gamma}_i(j) = \frac{1}{T}\sum_t \hat{e}_{it}\hat{e}_{it+j}$, where the summation $\sum_t$ is defined over $1 \le t, t+j \le T$. To define the estimators of the long-run variances λ_e^n, ω_e^2, and ϕ_e^4, we use the following kernel estimators of $\lambda_{e,i}$ and $\omega_{e,i}^2$,

$$\hat{\lambda}_{e,i} = \sum_{j=1}^{T-1} w\left(\frac{j}{h}\right)\hat{\Gamma}_i(j) \quad (12)$$

$$\hat{\omega}_{e,i}^2 = \sum_{j=-T+1}^{T-1} w\left(\frac{j}{h}\right)\hat{\Gamma}_i(j), \quad (13)$$

where $w(\cdot)$ is a kernel function and h is a bandwidth parameter. Define

$$\hat{\lambda}_e^n = \frac{1}{n}\sum_{i=1}^n \hat{\lambda}_{e,i}, \ \hat{\omega}_e^2 = \frac{1}{n}\sum_{i=1}^n \hat{\omega}_{e,i}^2, \ \text{and } \hat{\phi}_e^4 = \frac{1}{n}\sum_{i=1}^n \hat{\omega}_{e,i}^4. \quad (14)$$

In order for the estimators $\hat{\lambda}_e^n$, $\hat{\omega}_e^2$, and $\hat{\phi}_e^4$ to satisfy the desirable properties in $(9)-(11)$, we need the following assumptions on the kernel function and the bandwidth parameter.

Assumption 10 *(Restriction on the convergence rate of* n *and* T*). The size of the panel* (n,T) *tends to infinity with* $\liminf_{(n,T\to\infty)} \frac{\log T}{\log n} > 1$.

Define $a = \liminf_{(n,T\to\infty)} \frac{\log T}{\log n}$. The parameter a is related to the speed of $\frac{n}{T}$ tending to zero. The restriction $a > 1$ implies that $(n, T \to \infty)$ with $\frac{n}{T} \to 0$ because for n, T large,

$$\frac{n}{T} = e^{\log n - \log T} = e^{\left(1 - \frac{\log T}{\log n}\right)\log n} = n^{\left(1 - \frac{\log T}{\log n}\right)} \le n^{(1-a)} \to 0.$$

The above assumption allows the parameter a to be infinity.

Assumption 11 *(Kernel Conditions) The kernel function $w(\cdot) : \mathbb{R} \to [0,1]$ is continuous at zero and all but a finite number of other points, satisfying*

$$\text{(i) } w(0) = 1,\ \ w(x) = w(-x)\,,\ \ \int_{-\infty}^{\infty} w(x)^2\, dx < M,$$

$$\text{(ii) } w_q = \lim_{x\to 0} \left[1 - w(x) / |x|^q\right] < \infty$$

for some $0 < q \leq m$, where parameter m is defined in Assumptions 2 and 3.

Assumption 12 *(Kernel Conditions*) The kernel function $w(\cdot)$ satisfies the kernel conditions in Assumption 11 as well as*

$$\text{(iii) } \max\left\{1, \frac{1}{a-1}\right\} < q.$$

Assumption 13 *(Bandwidth Conditions) The bandwidths h_λ, h_ω, and h_ϕ tend to infinity satisfying the following conditions.*

(a) $h_\lambda \sim n^b$ with $\frac{1}{2q} < b < \min\left\{\frac{a-1}{2}, \frac{a}{q}, \frac{1}{2}\right\}$.

(b) For $0 < q < 1$, $h_\omega \sim n^b$ with $0 < b < \min\left\{1, \frac{a}{2}\right\}$. For $q \geq 1$, $h_\omega \sim n^b$ with $0 < b < \min\left\{1, \frac{a}{2}, \frac{a}{q}\right\}$.

(c) For $0 < q < 1$, $h_\phi \sim n^b$ with $0 < b < \frac{1}{4}$. For $q \geq 1$, $h_\phi \sim n^b$ with $0 < b < \min\left\{\frac{1}{4}, \frac{a}{q}\right\}$.

Lemma 4 *Suppose that Assumptions 1 – 10 hold.*

(a) If the kernel window satisfies Assumption 12 and the bandwidth h_λ satisfies Assumption 13(a), then,

$$\sqrt{n}\left(\hat{\lambda}_e^n - \lambda_e^n\right) = o_p(1).$$

(b) If the kernel window satisfies Assumption 11 and the bandwidth h_ω satisfies Assumption 13(b), then

$$\hat{\omega}_e^2 - \omega_e^2 = o_p(1)\,.$$

(c) If the kernel window satisfies Assumption 11 and the bandwidth h_ϕ satisfies Assumption 13(c), then

$$\hat{\phi}_e^4 - \phi_e^4 = o_p(1)\,.$$

In view of (6) and (7), using (8) and (14), we may define the following $t-$ statistics for the unit root null:

$$t_a^* = \frac{\sqrt{n}T\left(\hat{\rho}_{pool}^* - 1\right)}{\sqrt{\frac{2\hat{\phi}_e^4}{\hat{\omega}_e^4}}},$$

and

$$t_b^* = \sqrt{n}T\left(\hat{\rho}_{pool}^* - 1\right)\sqrt{\frac{1}{nT^2}tr\left(Z_{-1}Q_{\hat{\beta}_K}Z_{-1}'\right)}\left(\frac{\hat{\omega}_e}{\hat{\phi}_e^2}\right)$$

where

$$\hat{\rho}_{pool}^* = \frac{tr\left(Z_{-1}Q_{\hat{\beta}_K}Z'\right) - nT\hat{\lambda}_e^n}{tr\left(Z_{-1}Q_{\hat{\beta}_K}Z_{-1}'\right)}.$$

Theorem 2 *Suppose that Assumptions 1 – 13 hold. Then, under the null hypothesis,*

$$t_a^*,\ t_b^* \Rightarrow N\left(-\mu_\theta\sqrt{\frac{\omega_e^4}{2\phi_e^4}}, 1\right).$$

1.1.3 Estimation of the Number of Factors

In this section we discuss how to obtain a consistent estimator of the unknown number of factors, K. Now for a given $(n \times r)$ matrix β_r, let

$$\begin{aligned} W_{nT}(\beta_r, r) &= \min_{f_r} \frac{tr\left(\left(\hat{y} - f_r\beta_r'\right)\left(\hat{y} - f_r\beta_r'\right)'\right)}{nT} \\ &= \frac{tr\left(\hat{y}'Q_{\beta_r}\hat{y}\right)}{nT}. \end{aligned}$$

To estimate the true number of factors, K, Bai and Ng (2002) propose to maximize the following criterion function,

$$\begin{aligned} PC(r) &= W_{nT}\left(\hat{\beta}_r, r\right) + rG_{nT}, \\ IC(r) &= \ln\left(W_{nT}\left(\hat{\beta}_r, r\right)\right) + rG_{nT}, \end{aligned}$$

where the penalty function $G_{n,T}$ satisfies (i) $G_{n,T} \to 0$ and (ii) $\min\{n, T\} G_{n,T} \to \infty$. as $(n, T \to \infty)$.

Theorem 3 *Suppose that Assumptions 1 – 9 hold and $(n, T \to \infty)$ following Assumption 10. Let*

$$\hat{K} = \arg\min_{1 \le r \le \bar{K}} PC(r),\ \ \check{K} = \arg\min_{1 \le r \le \bar{K}} IC(r).$$

Then, under the null of unit root,

$$(a)\ plim1\left\{\hat{K} = K\right\} = 1 \text{ and } (b)\ plim1\left\{\check{K} = K\right\} = 1.$$

1.2 A Model with Incidental Trends

In this section, we extend our analysis and consider the dynamic panel model with deterministic trends:

$$\begin{aligned} z_{it} &= \alpha_{ki}'g_{kt} + z_{it}^0 \\ z_{it}^0 &= \rho_i z_{it-1}^0 + y_{it}, \end{aligned} \quad (15)$$

where

$$g_{0t} = 1 \text{ and } g_{1t} = (1, t)'.$$

We continue to assume the local-to-unity framework (2) for ρ_i and the approximate factor structure (3) for y_{it}. We also assume that $z_{i0}^0 = 0$ for all i. This model (15) is an extension of the model in the previous section as it adds incidental trend components $\alpha_{ki}' g_{kt}$ representing individual effects. When $k = 0$, *i.e.*, $g_{kt} = 1$, the model with incidental trends (15) reduces to our original model (1). In this case, assuming Assumption [individual effect], the panel unit root test statistic of the previous section was constructed ignoring the incidental parameter. In this section, we take into account the incidental parameters (or trends). We say the cases of $k = 0$ and $k = 1$ as model $k = 0$ and model $k = 1$, respectivley. As in the previous section, we want to test for the null hypothesis $\mathbb{H}_0$ against the (local) alternative $\mathbb{H}_1$.

The main purpose of this section is to study the local power of a test constructed using the t statistics based on a (bias-modified) pooled estimator such as $\hat{\rho}^+$. To simply the analysis, we assume that

$$\rho_i = 1 - \frac{\mu_\theta}{n^\eta T} \text{ for all } i,$$

$e_{it} \sim iid\,(0,1)$ with finite fourth moments across i and over t, $f_t \sim iid\,(0,1)$ over t, and e_{is} and f_t are independent. We also assume for convenience that the factor loading coefficient β_i^0 is observed. In what follows, we will investigate the asymptotic powers of the models $k = 0$ and $k = 1$ within a $\frac{1}{n^\eta T}-$ neighborhood of the null hypothesis of a unit root and find that the test has no asymptotic power if $\frac{1}{4} < \eta$.for $k = 0$ and $\frac{1}{6} < \eta$.for $k = 1$. The restrictions made in this section could be relaxed to the more general conditions assumed in the previous section without changing any of the main results.

Define

$$\hat{\rho}_{pool}^{\#} = \frac{tr\left(\tilde{Z}_{-1} Q_{\beta^0} \tilde{Z}'\right) - nT b_{k,nT},}{tr\left(\tilde{Z}_{-1} Q_{\beta^0} \tilde{Z}_{-1}'\right)}$$

where $b_{k,nT} = \frac{1}{nT} E\left(tr\left(\tilde{E}_{-1}\tilde{e}'\right)\right)$. The limit of $b_{kn,T}$ as $T \to \infty$ is $b_k = -E\left(\int_0^1 \int_0^1 W(r)\, h_k(r,s)\, dW(s)\, dr\right)$[2], where $W(r)$ is a Wiener process, $h_k(r,s) = g_k(r)' \left(\int_0^1 g_k(r)\, g_k(r)'\, dr\right)^{-1} g_k(s)$, $g_0(r) = 1$ and $g_1(r) = (1, r)'$. The correction term $b_{k,nT}$ is the mean of the bias due to the serial correlation generated by the detrended data $\tilde{E}_{-1}$ and $\tilde{e}$.

The typical $t-$ ratio statistic is defined as

$$t^{\#} = \sqrt{tr\left(\tilde{Z}_{-1} Q_{\beta^0} \tilde{Z}_{-1}'\right)}\left(\hat{\rho}_{pool}^{\#} - 1\right).$$

Lemma 5 *Assume that $\eta > \frac{1}{4}$ for model $k = 0$ and $\eta > \frac{1}{6}$ for model $k = 1$. Under the assumptions made in this section, the following hold.*

(a) $\frac{1}{nT^2} tr\left(\tilde{Z}_{-1}^0 Q_{\beta^0} \tilde{Z}_{-1}^{0\prime}\right) \to_p \left(\int_0^1 r dr - \int_0^1 \int_0^1 \min(r,s)\, h_k(r,s)\, ds dr\right)$.

(b) $\sqrt{n}\left[\frac{1}{nT} tr\left(\tilde{E}_{-1}\tilde{e}'\right) - b_{k,nT}\right] \Rightarrow N\left(0, \lim_{n,T} E\left(\frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \tilde{E}_{it-1}\tilde{e}_{it} - b_{k,nT}\right)^2\right)$.

(c) $n^{1/2-\eta}\left[\frac{1}{nT} tr\left(\left(\tilde{Z}_{-1}^0 - \tilde{Z}_{-1}^0(0)\right) Q_{\beta^0}\tilde{e}'\right) - \frac{\mu_\theta}{n^\eta}\int_0^1 \int_0^r (r-s)\, h_k(r,s)\, ds dr\right] = o_p(1)$.

[2] A direct calculation shows that $b_k = -\frac{1}{2}$ for $k = 0$ and 1.

(d) $\sqrt{n}\left[\frac{1}{nT^2}tr\left(\tilde{Z}_{-1}^0(0)\,Q_{\beta^0}\tilde{Z}_{-1}^{0\prime}(0)\right)-\left(\int_0^1 rdr-\int_0^1\int_0^1\min(r,s)\,h_k(r,s)\,dsdr\right)\right]=O_p(1)$.

(e) $\frac{1}{n^{1/2+\eta}T^2}tr\left(\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0(0)\right)Q_{\beta^0}\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0(0)\right)'\right)=o_p(1)$.

(f) $\frac{1}{n^{1/2+\eta}T^2}tr\left(\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0(0)\right)Q_{\beta^0}\tilde{Z}_{-1}'(0)\right)=o_p(1)$.

Theorem 4 *Assume that $\eta>\frac{1}{4}$ for model $k=0$ and $\eta>\frac{1}{6}$ for model $k=1$. Under the assumptions made in this section, the $t^{\#}$ statistic does not have an asymptotic power in a $\frac{1}{n^\eta T}$ neighborhood of the null of unit root.*

2 Omitted Proofs

2.1 Appendix A

Suppose that A and B are $(n\times n)$ matrices. The following facts will be used frequently in the following proofs; (a) $tr(AB)\leq\|A\|\,\|B\|$ by the Cauchy-Schwarz inequality, (b) if A is symmetric and positive semidefinite, then $\|A\|\leq tr(A)$ and $tr(A)\leq\sqrt{n}\,\|A\|$, and (c) if both of A and B are positive semidefinite, then $tr(AB)\leq tr(A)\,\|B\|$, and $tr(AB)\leq tr(B)\,\|A\|$. To distinguish the notation for the panel with $\rho_i=1(c_i=0)$ for all i, we denote $Z(0)$ and $Z^0(0)$ for Z and Z^0 in (4), respectively. Also we define $F^0=\Xi f^0$ and $E=\Xi e$, where Ξ be a $(T\times T)$ lower triangular matrix such that

$$\underset{(T\times T)}{\Xi}=\begin{pmatrix}1&0&\cdots&0\\1&1&&0\\\vdots&\vdots&\ddots&\\1&1&\cdots&1\end{pmatrix},$$

So,

$$\begin{aligned}Z(0)&=l_T\alpha'+Z^0(0),\\Z^0(0)&=F^0\beta^{0\prime}+E,\end{aligned}$$

where $l_T=\underbrace{(1,...,1)}_{T}'$. Similarly we define $Z_{-1},Z_{-1}(0),Z_{-1}^0(0)$, F_{-1}^0, and E_{-1} to denote the matrices of lagged panel data of $Z,Z(0),Z^0(0)$, F^0, and E, respectively.

2.1.1 Properties of e_{it}

A BN-Decomposition of e_{it}

We first introduce the BN-decomposition of the linear process e_{it} (see Phillips and Solo, 1992 for more details). Under Assumptions 2, it is possible to decompose e_{it} into

$$e_{it}=d_i(1)\,v_{it}+\breve{e}_{it-1}-\breve{e}_{it},\tag{16}$$

where

$$d_i(1)=\sum_{j=0}^{\infty}d_{ij},\;\;\breve{e}_{it}=\sum_{j=0}^{\infty}\breve{d}_{ij}v_{it-j},\;\;\breve{d}_{ij}=\sum_{s=j+1}^{\infty}d_{is}.$$

B The following lemma holds under Assumption 2.

Lemma 6 *(Existence of moments of e_{it} and $\breve{e}_{it}$)*
(a) $\sup_{i,t} Ee_{it}^4 < M$,
(b) $\sup_{i,t} E\breve{e}_{it}^4 < M$,
(c) $sup_{i,T} E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(e_{it}^2 - \sigma_{e,i}^2\right)\right)^2 < M.$
(d) $sup_i E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_{it}\right)^4 < M.$

C $X_{ij,T} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(e_{it}e_{jt} - E\left(e_{it}e_{jt}\right)\right)$. The following lemma holds under Assumption 2.

Lemma 7 $sup_{i,j} E\left(X_{ij,T}^4\right) < M.$

Proof of Lemma 6
Part (a).
Notice that for some constant M_1,

$$\begin{aligned}\sup_{i,t} Ee_{it}^4 &= \sup_{i,t} E\left(\sum_{j=0}^{\infty} d_{ij}v_{it-j}\right)^4 \leq M_1\left(\sup_i \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} d_{ij}^2 d_{ik}^2 + \sup_i \sum_{j=0}^{\infty} d_{ij}^4\right) \\ &\leq M_1\left(\left(\sum_{j=0}^{\infty} \bar{d}_j^2\right)^2 + \left(\sum_{j=0}^{\infty} \bar{d}_j^4\right)\right).\end{aligned}$$

Since $\sum_{j=0}^{\infty} \bar{d}_j < M$,

$$\sum_{j=0}^{\infty} \bar{d}_j^2,\ \sum_{j=0}^{\infty} \bar{d}_j^4 < M,$$

and the proof is done. ■

Part (b).
Recall that

$$\breve{e}_{it} = \sum_{j=0}^{\infty} \breve{d}_{ij}v_{it-j}.$$

The required result follows because

$$\sum_{j=0}^{\infty}\sup_i \left|\breve{d}_{ij}\right| \leq \sum_{j=0}^{\infty}\sum_{s=j+1}^{\infty} \bar{d}_s = \sum_{j=0}^{\infty} j\bar{d}_j < M,$$

and so,

$$\sum_{j=0}^{\infty}\sup_i \left|\breve{d}_{ij}^k\right| < M$$

for any $k \geq 1$. ■

Part (c).

Notice that $\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(e_{it}^2-\sigma_{e,i}^2\right)\right)^2=X_{ii,T}^2$. Thus, Part (c) follows since $\sup_{i,j}EX_{ij,T}^4<M$. ■

Part (d).

By definition,

$$
\begin{aligned}
& E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^4 \\
=\ & \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{p=1}^{T}\sum_{q=1}^{T}E\left(e_{it}e_{is}e_{ip}e_{iq}\right) \\
=\ & \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{p=1}^{T}\sum_{q=1}^{T}\sum_{x}\sum_{y}\sum_{z}\sum_{w}d_{ix}d_{iy}d_{iz}d_{iw}E\left(v_{it-x}v_{is-y}v_{ip-z}v_{iq-w}\right). \quad (17)
\end{aligned}
$$

Four case: (i) $t-x=s-y\neq p-z=q-w$, (ii) $t-x=p-z\neq s-y=q-w$, (iii) $t-x=q-w\neq s-y=p-z$, (iv) $t-x=s-y=p-z=q-w$.

Case (i) $x=t-s+y$, $z=p-q+w$. So,

$$
(17)\leq M_1\sigma_u^4\sup_i\left(\frac{1}{T}\sum_{t\geq s=1}^{T}\sum_{y}d_{it-s+y}d_{iy}\right)^2\leq M_1\sigma_u^4\left(\sum_{y}\bar{d}_y\right)^2=M.
$$

Similarly, we can bound cases (ii) and (iii). For case (iv), $z=p-q+w$, $y=w+(p-q)+(s-q)$, $x=w+(p-q)+(s-q)+(t-s)$.

$$
\begin{aligned}
(17)\ \leq\ & M_2E\left(v_{it}^4\right)\left(\frac{1}{T^2}\sum_{q<p<s<t}\sum_{w}d_{iw}d_{iw+(p-q)}d_{iw+(p-q)+(s-q)}d_{iw+(p-q)+(s-p)+(t-s)}\right) \\
\leq\ & M_3\left(\frac{1}{T^2}\sum_{w=0}^{\infty}\sum_{h=0}^{T-1}\sum_{m=0}^{T-h-1}\sum_{n=0}^{T-h-m-1}\sum_{t=h+m+n+1}^{T}\bar{d}_w\bar{d}_{w+h}\bar{d}_{w+h+m}\bar{d}_{w+h+m+n}\right) \\
=\ & M_3\left(\frac{1}{T}\sum_{w=0}^{\infty}\sum_{h=0}^{T-1}\sum_{m=0}^{T-h-1}\sum_{n=0}^{T-h-m-1}\left(\frac{T-h-m-n-1}{T}\right)\bar{d}_w\bar{d}_{w+h}\bar{d}_{w+h+m}\bar{d}_{w+h+m+n}\right) \\
\leq\ & \frac{1}{T}M_3\left(\sum_{l}\bar{d}_l\right)^4<\frac{M}{T}. \blacksquare
\end{aligned}
$$

Proof of Lemma 7

Recall that

$$
X_{ij,T}=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(e_{it}e_{jt}-E\left(e_{it}e_{jt}\right)\right)
$$

We first introduce the BN-Decomposition of $e_{it}e_{jt}$ (for more details on this, refer to Phillips and Solo, 1992). Notice that

$$
\begin{aligned}
e_{it}e_{jt}\ =\ & \left(\sum_{k=0}^{\infty}d_{ik}v_{it-k}\right)\left(\sum_{l=0}^{\infty}d_{jl}v_{jt-l}\right) \\
=\ & \sum_{k=0}^{\infty}d_{ik}d_{jk}v_{it-k}v_{jt-k}+\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}d_{ik}d_{jk+m}v_{it-k}v_{jt-k-m}+\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}d_{jk}d_{ik+m}v_{jt-k}v_{it-k-m}.
\end{aligned}
$$

Define

$$D_{ij}^{m}(L)=\sum_{k=0}^{\infty}D_{ij,k}^{m}L^{k},\ \ D_{ij,k}^{m}=d_{ik}d_{jk+m}$$

and

$$D_{ji}^{m}(L)=\sum_{k=0}^{\infty}D_{ji,k}^{m}L^{k},\ \ D_{ji,k}^{m}=d_{jk}d_{ik+m}.$$

Then, we write

$$e_{it}e_{jt}=D_{ij}^{0}(L)\,v_{it}v_{jt}+\sum_{m=1}^{\infty}D_{ij}^{m}(L)\,v_{it}v_{jt-m}+\sum_{m=1}^{\infty}D_{ji}^{m}(L)\,v_{jt}v_{it-m}.$$

As shown by Phillips and Solo (1992), under the assumption, we can decompose $D_{ij}^{m}(L)$ as follows,

$$\begin{aligned}D_{ij}^{m}(L) &= D_{ij}^{m}(1)+(1-L)\,\breve{D}_{ij}^{m}(L)\,,\\ \breve{D}_{ij}^{m}(L) &= \sum_{k=0}^{\infty}\breve{D}_{ij,k}^{m}L^{k},\ \ \breve{D}_{ij,k}^{m}=\sum_{l=k+1}^{\infty}D_{ij,l}^{m}=\sum_{l=k+1}^{\infty}d_{il}d_{jl+m}.\end{aligned}$$

Similarly, we can decompose $D_{ji}^{m}(L)$, too. Using this, then, we write

$$\begin{aligned}e_{it}e_{jt} &= D_{ij}^{0}(1)\,v_{it}v_{jt}+(1-L)\,\breve{D}_{ij}^{0}(L)\,v_{it}v_{jt}\\ &\quad+\sum_{m=1}^{\infty}D_{ij}^{m}(1)\,v_{it}v_{jt-m}+(1-L)\sum_{m=1}^{\infty}\breve{D}_{ij}^{m}(L)\,v_{it}v_{jt-m}\\ &\quad+\sum_{m=1}^{\infty}D_{ji}^{m}(1)\,v_{jt}v_{it-m}+(1-L)\sum_{m=1}^{\infty}\breve{D}_{ji}^{m}(L)\,v_{jt}v_{it-m},\end{aligned}$$

and

$$\begin{aligned}X_{ij,T} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(D_{ij}^{0}(1)\,v_{it}v_{jt}-D_{ii}^{0}(1)\right)+\frac{1}{\sqrt{T}}\left(\breve{D}_{ij}^{0}(L)\,v_{i1}v_{j1}-\breve{D}_{ij}^{0}(L)\,v_{iT}v_{jT}\right)\\ &\quad+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\sum_{m=1}^{\infty}D_{ij}^{m}(1)\,v_{it}v_{jt-m}\right)+\frac{1}{\sqrt{T}}\left(\sum_{m=1}^{\infty}\breve{D}_{ij}^{m}(L)\,v_{i1}v_{j1-m}-\sum_{m=1}^{\infty}\breve{D}_{ij}^{m}(L)\,v_{iT}v_{jT-m}\right)\\ &\quad+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\sum_{m=1}^{\infty}D_{ji}^{m}(1)\,v_{jt}v_{it-m}\right)+\frac{1}{\sqrt{T}}\left(\sum_{m=1}^{\infty}\breve{D}_{ji}^{m}(L)\,v_{j1}v_{i1-m}-\sum_{m=1}^{\infty}\breve{D}_{ji}^{m}(L)\,v_{jT}v_{iT-m}\right).\end{aligned}$$

To show that $\sup_{ij}EX_{ij,T}^{4}<M$, it is enough to show that

$$\sup_{ij}E(X_{k,ij,T}^{4})<M,\ \ k=1,...,4,$$

where

$$
\begin{aligned}
X_{1,ij,T} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(D_{ij}^{0}(1)\,v_{it}v_{jt}-D_{ii}^{0}(1)\right)\\
X_{2,ij,T} &= \frac{1}{\sqrt{T}}\breve{D}_{ij}^{0}(L)\,v_{it}v_{jt}\\
X_{3,ij,T} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\sum_{m=1}^{\infty}D_{ij}^{m}(1)\,v_{it}v_{jt-m}\right)\\
X_{4,ij,T} &= \frac{1}{\sqrt{T}}\sum_{m=1}^{\infty}\breve{D}_{ij}^{m}(L)\,v_{it}v_{jt-m}.
\end{aligned}
$$

1. (i) When $i=j$: under the assumption (i),

$$
\begin{aligned}
\sup_{i} EX_{1,ii,T}^{4} &= E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(v_{it}^{2}-1\right)\right)^{4}D_{ii}^{0}(1)^{4}\\
&\leq D^{0}(1)^{4}\left(3\kappa_{4}^{2}+\frac{\kappa_{8}}{T}\right)<M,
\end{aligned}
$$

because

$$
D^{0}(1)=\sum_{k=0}^{\infty}\bar{d}_{k}^{2}<M
$$

under the assumption (ii).
(ii) When $i\neq j$:

$$
\begin{aligned}
\sup_{\substack{i,j\\ i\neq j}} EX_{1,ij,T}^{4} &= \sup_{\substack{i,j\\ i\neq j}} D_{ij}^{0}(1)^{4}\,E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}v_{it}v_{jt}\right)^{4}\\
&\leq D^{0}(1)^{4}\left(3+\frac{\kappa_{4}^{2}}{T}\right)<M.
\end{aligned}
$$

Thus,

$$
\sup_{ij} EX_{1,ij,T}^{4}<M.
$$

2. For some finte constant M_1, we have

$$
\begin{aligned}
\sup_{ij} EX_{2,ij,T}^{4} &= \frac{1}{T^{2}}\sup_{ij} E\left(\breve{D}_{ij}^{0}(L)\,v_{it}v_{jt}\right)^{4}=\frac{1}{T^{2}}\sup_{ij} E\left(\sum_{k=0}^{\infty}\breve{D}_{ij,k}^{0}v_{it-k}v_{jt-k}\right)^{4}\\
&= \frac{M_{1}}{T^{2}}\left(\sup_{ij}\left(\sum_{k=0}^{\infty}\left(\breve{D}_{ij,k}^{0}\right)^{2}\right)^{2}+\sup_{ij}\left(\sum_{k=0}^{\infty}\left(\breve{D}_{ij,k}^{0}\right)^{4}\right)\right).
\end{aligned}
$$

Under the assumption (ii),

$$
\begin{aligned}
\sup_{ij}\sum_{k=0}^{\infty}\left(\breve{D}_{ij,k}^{0}\right)^{2} &= \sup_{ij}\sum_{k=0}^{\infty}\left(\sum_{l=k+1}^{\infty}d_{il}d_{jl}\right)^{2}\leq\sum_{k=0}^{\infty}\left(\sum_{l=k+1}^{\infty}\bar{d}_{l}^{2}\right)^{2}\leq\left(\sum_{k=0}^{\infty}\sum_{l=k+1}^{\infty}\bar{d}_{l}^{2}\right)^{2}\\
&= \left(\sum_{l=1}^{\infty}l\bar{d}_{l}^{2}\right)^{2}<M.
\end{aligned}
$$

Similarly,

$$\sup_{ij}\sum_{k=0}^{\infty}\left(\breve{D}_{ij,k}^{0}\right)^{4}\leq\sum_{k=0}^{\infty}\left(\sum_{l=k+1}^{\infty}\bar{d}_{l}^{2}\right)^{4}\leq\left(\sum_{l=1}^{\infty}l\bar{d}_{l}^{2}\right)^{4}<M.$$

Thus,

$$\sup_{ij}EX_{2,ij,T}^{4}<\frac{M}{T^{2}}\rightarrow 0.$$

3. Define

$$v_{ijt}^{m}=\sum_{m=1}^{\infty}D_{ij}^{m}\left(1\right)v_{jt-m}.$$

For some finite constant M_1,

$$\sup_{ij}E\left(v_{ijt}^{m}\right)^{4}=\sup_{ij}E\left(\sum_{m=1}^{\infty}D_{ij}^{m}\left(1\right)v_{jt-m}\right)^{4}\leq M_{1}\sup_{ij}\left[\left(\sum_{m=1}^{\infty}D_{ij}^{m}\left(1\right)^{2}\right)^{2}+\left(\sum_{m=1}^{\infty}D_{ij}^{m}\left(1\right)^{4}\right)\right].$$

Since

$$\sup_{ij}\left(\sum_{m=1}^{\infty}\left|D_{ij}^{m}\left(1\right)\right|^{l}\right)\leq\sum_{m=1}^{\infty}\left(\sum_{k=0}^{\infty}\bar{d}_{k}\bar{d}_{k+l}\right)^{l}\leq\left(\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\bar{d}_{k}\bar{d}_{k+l}\right)^{l}\leq\left(\sum_{k=0}^{\infty}\bar{d}_{k}\right)^{2l}<M$$

for any integer >0.

Now, for some constant M_1,

$$\begin{aligned}\sup_{ij}EX_{3,ij,T}^{4} &= \sup_{ij}E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}v_{it}v_{ijt}^{m}\right)^{4}\\ &= \sup_{ij}\frac{1}{T^{2}}\sum_{t_1}\sum_{t_2}\sum_{t_3}\sum_{t_4}E\left(v_{it_1}v_{ijt_1}^{m}v_{it_2}v_{ijt_2}^{m}v_{it_3}v_{ijt_3}^{m}v_{it_4}v_{ijt_4}^{m}\right)\\ &= 3\sup_{ij}\frac{1}{T^{2}}\sum_{t_1}\sum_{\substack{t_2\\ t_1\neq t_2}}E\left(v_{it_1}^{2}\left(v_{ijt_1}^{m}\right)^{2}v_{it_2}^{2}\left(v_{ijt_2}^{m}\right)^{2}\right)+\sup_{ij}\frac{1}{T^{2}}\sum_{t}E\left(v_{it}^{4}\left(v_{ijt}^{m}\right)^{4}\right)\\ &\leq M_{1}\sup_{ij}\left(\frac{1}{T}\sum_{t}\sqrt{E\left(v_{it}^{4}\left(v_{ijt}^{m}\right)^{4}\right)}\right)^{2}=M_{1}\kappa_{4}\left(\frac{1}{T}\sum_{t}\sqrt{\sup_{ij}E\left(v_{ijt}^{m}\right)^{4}}\right)^{2}<M,\end{aligned}$$

where the last equality holds because v_{it} and v_{ijt}^{m} are uncorrelated.

4. Before we start, we calculate two moments. Define

$$\breve{v}_{ij,t-l}^{m}=\sum_{m=1}^{\infty}\breve{D}_{ij,l}^{m}v_{jt-l-m}.$$

For some finite constant M_1,

$$\sup_{ij}\sum_{l=0}^{\infty}E\left(\breve{v}_{ij,t-l}^{m}\right)^{4}\leq M_{1}\sup_{ij}\left(\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\left(\breve{D}_{ij,l}^{m}\right)^{4}+\sum_{l=0}^{\infty}\left(\sum_{m=1}^{\infty}\left(\breve{D}_{ij,l}^{m}\right)^{2}\right)^{2}\right).$$

Notice

$$
\begin{aligned}
\sup_{ij}\left(\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\left(\breve{D}_{ij,l}^{m}\right)^{4}\right) &= \sup_{ij}\left(\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}d_{ik}d_{jk+m}\right)^{4}\right) \leq \sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{4} \\
&\leq \sum_{l=0}^{\infty}\left(\sum_{m=1}^{\infty}\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{4} \leq \sum_{l=0}^{\infty}\left(\sum_{k=l+1}^{\infty}\bar{d}_{k}\right)^{8} \leq \left(\sum_{l=0}^{\infty}\sum_{k=l+1}^{\infty}\bar{d}_{k}\right)^{8} \\
&= \left(\sum_{k=0}^{\infty}k\bar{d}_{k}\right)^{8} < M.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\sup_{ij}\sum_{l=0}^{\infty}\left(\sum_{m=1}^{\infty}\left(\breve{D}_{ij,l}^{m}\right)^{2}\right)^{2} &= \sup_{ij}\sum_{l=0}^{\infty}\left(\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}d_{ik}d_{jk+m}\right)^{2}\right)^{2} \leq \sum_{l=0}^{\infty}\left(\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{2}\right)^{2} \\
&\leq \sum_{l=0}^{\infty}\left(\sum_{m=1}^{\infty}\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{4} \leq \left(\sum_{k=0}^{\infty}k\bar{d}_{k}\right)^{8} < M.
\end{aligned}
$$

So,

$$
\sup_{ij}\sum_{l=0}^{\infty}E\left(\breve{v}_{ij,t-l}^{m}\right)^{4} < M. \tag{18}
$$

Next, for some finite constant M_1,

$$
\begin{aligned}
&\sup_{ij}\sum_{\substack{l_1=0\\ l_1\neq l_2}}^{\infty}\sum_{l_2=0}^{\infty}E\left[v_{it-l_1}^{2}\left(\breve{v}_{ij,t-l_1}^{m}\right)^{2}\right]\left[v_{it-l_2}^{2}\left(\breve{v}_{ij,t-l_2}^{m}\right)^{2}\right] \\
\leq\; &\sup_{ij}\left(\sum_{l=0}^{\infty}\sqrt{E\left[v_{it-l}^{4}\left(\breve{v}_{ij,t-l}^{m}\right)^{4}\right]}\right)^{2} = \kappa_4\sup_{ij}\left(\sum_{l=0}^{\infty}\sqrt{E\left(\breve{v}_{ij,t-l}^{m}\right)^{4}}\right)^{2},
\end{aligned}
$$

because v_{it-l} and $\breve{v}_{ij,t-l}^{m}$ are uncorrelated. Again, for some finite constant M_1,

$$
\begin{aligned}
\sup_{ij}\sum_{l=0}^{\infty}\sqrt{E\left(\breve{v}_{ij,t-l}^{m}\right)^{4}} &\leq M_1\sup_{ij}\sum_{l=0}^{\infty}\sqrt{\sum_{m=1}^{\infty}\left(\breve{D}_{ij,l}^{m}\right)^{4}+\left(\sum_{m=1}^{\infty}\left(\breve{D}_{ij,l}^{m}\right)^{2}\right)^{2}} \\
&\leq M_1\sum_{l=0}^{\infty}\sqrt{\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{4}}+\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{2} \\
&\leq 2M_1\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\left(\sum_{k=l+1}^{\infty}\bar{d}_{k}\bar{d}_{k+m}\right)^{2} \leq 2M_1\left(\sum_{k=0}^{\infty}k\bar{d}_{k}\right)^{4} < M.
\end{aligned}
$$

Thus,

$$
\sup_{ij}\sum_{\substack{l_1=0\\ l_1\neq l_2}}^{\infty}\sum_{l_2=0}^{\infty}E\left[v_{it-l_1}^{2}\left(\breve{v}_{ij,t-l_1}^{m}\right)^{2}\right]\left[v_{it-l_2}^{2}\left(\breve{v}_{ij,t-l_2}^{m}\right)^{2}\right] < M. \tag{19}
$$

Now, for some finite constant M_1,

$$\begin{aligned}
\sup_{ij} EX_{4,ij,T}^4 &= \frac{1}{T^2}\sup_{ij} E\left(\sum_{m=1}^{\infty} \breve{D}_{ij}(L)\, v_{it}v_{jt-m}\right)^4 = \frac{1}{T^2}\sup_{ij} E\left(\sum_{m=1}^{\infty}\sum_{l=0}^{\infty} \breve{D}_{ij,l}^{m} v_{it-l}v_{jt-m-l}\right)^4 \\
&= \frac{1}{T^2}\sup_{ij} E\left(\sum_{l=0}^{\infty} v_{it-l}v_{ij,t-l}^{m}\right)^4 \\
&\leq \frac{M_1}{T^2}\sup_{ij}\left[\sum_{\substack{l_1=0\\ l_1\neq l_2}}^{\infty}\sum_{l_2=0}^{\infty} E\left[v_{it-l_1}^2\left(\breve{v}_{ij,t-l_1}^{m}\right)^2\right]\left[v_{it-l_2}^2\left(\breve{v}_{ij,t-l_2}^{m}\right)^2\right] + \sum_{l=0}^{\infty} E\left[v_{it-l}^4\left(\breve{v}_{ij,t-l}^{m}\right)^4\right]\right] \\
&\leq \frac{M_1\kappa_4}{T^2}\sup_{ij}\sum_{l=0}^{\infty} E\left(\breve{v}_{ij,t-l}^{m}\right)^4 + \frac{M_1}{T^2}\sup_{ij}\left[\sum_{\substack{l_1=0\\ l_1\neq l_2}}^{\infty}\sum_{l_2=0}^{\infty} E\left[v_{it-l_1}^2\left(\breve{v}_{ij,t-l_1}^{m}\right)^2\right]\left[v_{it-l_2}^2\left(\breve{v}_{ij,t-l_2}^{m}\right)^2\right]\right] \\
&< M,
\end{aligned}$$

where the final inequality holds by (18) and (19). ■

2.1.2 Preliminary Results

Lemma 8 *Under Assumptions 1 – 9, the following hold. Let* $E_{it} = \sum_{s=1}^{t} e_{is}$ *with* $E_{i0} = 0$.

(a) As $(n, T \to \infty)$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\theta_i E_{it-1}^2 \to_p \frac{1}{2}\mu_\theta\omega_e^2$.

(b) As $(n, T \to \infty)$ *with* $\frac{n}{T} \to 0$, $\sqrt{n}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E_{it-1}e_{it} - \lambda_e^n\right) \Rightarrow N\left(0, \frac{1}{2}\phi_e^4\right)$.

Lemma 9 *We assume Assumptions 1 – 9. Then, Parts (h) holds as* $(n, T \to \infty)$ *with* $\frac{n}{T} \to 0$, *and the other parts hold as* $(n, T \to \infty)$, *where*

(a) $\frac{1}{nT^2}\left\|Z_{-1}\right\|^2 = O_p(1)$,

(b) $\frac{1}{nT}\left\|Z_{-1}'y + y'Z_{-1}\right\| = O_p(1)$,

(c) $\frac{1}{nT^2} tr\left(\beta^{0\prime}E_{-1}'E_{-1}\beta^0\right) = O_p(1)$,

(d) $\frac{1}{nT}\left\|\beta^{0\prime}e'E_{-1}\beta^0 + \beta^{0\prime}E_{-1}'e\beta^0\right\| = O_p(1)$,

(e) $\frac{1}{nT}\left\|e'e\right\| = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{n}}\right)\right)$,

(f) $\frac{1}{nT} tr\left(f^{0\prime}ee'f^0\right) = O_p(1)$,

(g) $\frac{1}{nT} tr\left(\beta^{0\prime}e'e\beta^0\right) = O_p(1)$,

(h) $\frac{1}{n\sqrt{n}T\sqrt{T}}\left(\beta^{0\prime}e'eE_T\right) = o_p(1)$, *where* $E_T = e'l_T$,

(i) $\frac{1}{\sqrt{n}\sqrt{T}}\left\|\beta^{0\prime}e'l_T\right\| = O_p(1)$. *(i*)* $\frac{1}{\sqrt{n}\sqrt{T}}\left\|e'l_T\right\| = O_p(1)$, *(i**)* $\frac{1}{\sqrt{n}\sqrt{T}}\left\|y'l_T\right\| = O_p(1)$.

(j) $\frac{1}{\sqrt{n}T\sqrt{T}}\left\|\beta^{0\prime}E_{-1}'l_T\right\| = O_p(1)$,

(k) $\frac{\|y\|^2}{nT} = O_p(1)$,

(l) $\frac{1}{nT\sqrt{T}}\left|\alpha'E_{-1}'l_T\right| = O_p(1)$.

Lemma 10 *Suppose that Assumptions 1 – 9 hold. Then, the following hold.*

(a) $\frac{1}{nT^2}\sum_{i=1}^{n}\theta_i^2\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\frac{t-s-1}{T}\beta_i^{0\prime}f_s^0\right)^2 = O_p(1)$.

(b) $\frac{1}{nT^2}\sum_{i=1}^{n}\theta_i^2\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\frac{t-s-1}{T}e_{is}\right)^2 = O_p(1)$.

(c) $\frac{1}{nT}\sum_{i=1}^{n}\theta_i\beta_i^{0\prime}\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{t-s-1}{T}f_s^0 f_t^{0\prime}\right)\beta_i^0 = O_p(1)$.

(d) $\frac{1}{nT}\sum_{i=1}^{n}\theta_i\beta_i^{0\prime}\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{t-s-1}{T}f_s^0 e_{it}\right) = o_p(1)$.

(e) $\frac{1}{nT}\sum_{i=1}^{n}\theta_i\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\frac{t-s-1}{T}e_{is}e_{it}\right) = o_p(1)$.

(f) $\left\|\frac{1}{n^2T}\sum_{i=1}^{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\theta_j\sum_{t=2}^{T}e_{it}\left(\sum_{s=1}^{t-1}\frac{t-s-1}{T}f_s^0\right)\beta_j^{0\prime}\right\| = o_p(1)$

(g) $\left\|\frac{1}{n^2T}\sum_{i=1}^{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\theta_j\sum_{t=2}^{T}e_{it}\left(\sum_{s=1}^{t-1}\frac{t-s-1}{T}e_{js}\right)\beta_j^{0\prime}\right\| = o_p(1)$.

(h) $\frac{1}{n\sqrt{T}}\left\|\sum_{i=1}^{n}\theta_i\beta_i^0\beta_i^{0\prime}\left(\sum_{s=1}^{T}\left(1-\frac{s}{T}\right)f_s^0\right)\right\| = O_p(1)$.

(i) $\frac{1}{n\sqrt{T}}\left\|\sum_{i=1}^{n}\beta_i^0\theta_i\left(\sum_{s=1}^{T}\left(1-\frac{s}{T}\right)e_{is}\right)\right\| = o_p(1)$.

Lemma 11 *Suppose that Assumptions 1 – 9 hold. Then, the following hold.*

(a) $\frac{1}{T}\left\|Z_{-1}-Z_{-1}(0)\right\| = O_p(1)$.

(b) $\frac{1}{\sqrt{nT}}tr\left(\left(Z_{-1}-Z_{-1}(0)\right)y'\right) = O_p(1)$.

(c) $\frac{1}{\sqrt{nT}}tr\left(\left(Z_{-1}-Z_{-1}(0)\right)Q_{\beta_0}e'\right) = o_p(1)$.

(d) $\frac{1}{n\sqrt{T}}\left\|\beta^{0\prime}\left(Z_T-Z_T(0)\right)\right\|$,

where $Z_T=(z_{1T},...,z_{nT})'$ *and* $Z_T(0)=(z_{1T}(0),...,z_{nT}(0))'$.

Let β_r denote an $(n\times r)$ matrix, $r\le K$. Define

$$\mathcal{H}_{1nT}(\beta_r) = tr\left(\frac{\beta_r'}{\sqrt{n}}\left(\frac{\hat{y}'\hat{y}}{nT}\right)\frac{\beta_r}{\sqrt{n}}\right),$$

$$\mathcal{H}_{2nT}(\beta_r) = tr\left(\frac{\beta_r'}{\sqrt{n}}\left(\frac{y'y}{nT}\right)\frac{\beta_r}{\sqrt{n}}\right),$$

and

$$\mathcal{H}_{3nT}(\beta_r) = tr\left(\frac{\beta_r'}{\sqrt{n}}\left(\frac{\beta^0 f^{0\prime}f^0\beta^{0\prime}}{nT}\right)\frac{\beta_r}{\sqrt{n}}\right).$$

The following lemma establishes the uniform convergence of the three functions.

Lemma 12 *Suppose that Assumptions 1 – 9 hold. As* $(n,T\to\infty)$ *with* $\frac{n}{T}\to 0$,

(a) $\sup_{\frac{\beta_r'\beta_r}{n}=I_r}\left|\mathcal{H}_{1nT}(\beta_r)-\mathcal{H}_{2nT}(\beta_r)\right| = o_p(1)$

(b) $\sup_{\frac{\beta_r'\beta_r}{n}=I_r}\left|\mathcal{H}_{2nT}(\beta_r)-\mathcal{H}_{3nT}(\beta_r)\right| = o_p(1)$.

The relationship among the various estimators for β^0 and f^0 are well known. Let $\check{\Lambda}_{nT,K}$ denote the diagonal matrix of the K largest eigenvalues of $\hat{y}\hat{y}'$. Then, by definition,

$$\hat{y}\hat{y}'\frac{\check{f}_K}{\sqrt{T}} = \frac{\check{f}_K}{\sqrt{T}}\check{\Lambda}_{nT,K},$$

and so

$$\hat{y}'\hat{y}\left(\hat{y}'\frac{\check{f}_K}{\sqrt{T}}\check{\Lambda}_{nT,K}^{-1/2}\right) = \left(\hat{y}'\frac{\check{f}_K}{\sqrt{T}}\check{\Lambda}_{nT,K}^{-1/2}\right)\check{\Lambda}_{nT,K}.$$

Since $tr\left(\check{\Lambda}_{nT,K}^{-1/2}\frac{\check{f}_K'}{\sqrt{T}}\hat{y}\hat{y}'\frac{\check{f}_K}{\sqrt{T}}\check{\Lambda}_{nT,K}^{-1/2}\right) = I_K$, we have

$$\bar{\beta}_K = \frac{\sqrt{n}}{\sqrt{T}}\hat{y}'\check{f}_K\check{\Lambda}_{nT,K}^{-1/2},$$

and in consequence,

$$\bar{f}_K = \frac{1}{n}\hat{y}\bar{\beta}_K = \check{f}_K \left(\frac{\check{\Lambda}_{nT,K}}{nT}\right)^{1/2}.$$

Also, using the definition of $\check{\beta}_K = \frac{1}{T}\hat{y}'\check{f}_K$ and the relations above, we deduce that

$$\hat{\beta}_K = \check{\beta}_K \left(\frac{\check{\beta}_K'\check{\beta}_K}{n}\right)^{1/2} = \frac{1}{T}\hat{y}'\check{f}_K \left(\frac{\check{\Lambda}_{nT,K}}{nT}\right)^{1/2} = \frac{1}{T}\hat{y}'\bar{f}_K = \frac{\hat{y}'\hat{y}}{nT}\bar{\beta}_K.$$

This relation between $\hat{\beta}_K$ and $\bar{\beta}_K$ will be used a lot in the proofs of the appendix.

Recall that $\bar{\beta}_K$ is $\sqrt{n}$ times the $(n \times K)$ matrix of the orthonormal eigenvectors of the first K largest eigenvalues of $\frac{\hat{y}'\hat{y}}{nT}$. Let $\Lambda_{nT,K}$ be the $(K \times K)$ diagonal matrix consisting of the first K largest eigenvalues of $\frac{\hat{y}'\hat{y}}{nT}$ (and also of $\hat{y}'\hat{y}$), *i.e.*,

$$\frac{\hat{y}'\hat{y}}{nT}\bar{\beta}_K = \bar{\beta}_K\Lambda_{nT,K}.$$

Define Λ_K to the $(K \times K)$ diagonal matrix consisting of the eigenvalues of $\Sigma_f\Sigma_\beta$. The following lemma shows that the limit of $\Lambda_{nT,K}$ is Λ_K. This lemma corresponds to Lemma A.3 of Bai (2001), which was also implicitly proved by Stock and Watson (1998). The main difference between the two lemmas is that Bai analyzes the relationship between two estimators of the factors f_t^0 using the observable data, while the following lemma characterizes the relationship between two estimators of the factor loadings β_i^0 using the residuals.

Lemma 13 *As $(n, T \to \infty)$ with $\frac{n}{T} \to 0$, under Assumptions 1 – 9, the following hold.*

(a) $\frac{1}{n}\bar{\beta}_K'\frac{\hat{y}'\hat{y}}{nT}\bar{\beta}_K = \Lambda_{nT,K} \to_p \Lambda_K$.

(b) $\left(\frac{\bar{\beta}_K'\beta^0}{n}\right)\left(\frac{f^{0\prime}f^0}{n}\right)\left(\frac{\beta^{0\prime}\bar{\beta}_K}{n}\right) \to_p \Lambda_K$.

Lemma 14 *Suppose that Assumptions 1 – 9 hold. Assume that $(n, T \to \infty)$ with $\frac{n}{T} \to 0$. Then, the limit of*

$$H_K = \left(\frac{f^{0\prime}f^0}{T}\right)\left(\frac{\beta^{0\prime}\bar{\beta}_K}{n}\right)$$

is of full rank, and H_K is asymptotically bounded.

Lemma 15 *Suppose that Assumptions 1 – 9 hold.*

(a) Suppose that $(n, T \to \infty)$. Then,

$$\left\|\frac{\hat{\beta}_K - \beta_K^*}{\sqrt{n}}\right\| = O_p\left(\max\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right)\right),$$

where $\beta_K^* = \beta^0 H_K$, $H_K = \left(\frac{f^{0\prime}f^0}{n}\right)\left(\frac{\beta^{0\prime}\bar{\beta}_K}{n}\right)$.

(b) Suppose that $(n, T \to \infty)$ with $\frac{n}{T} \to 0$. Then,

$$\left\|\hat{\beta}_K - \beta_K^*\right\| = O_p(1).$$

(c) Suppose that $(n, T \to \infty)$ with $\frac{n}{T} \to 0$. Then,

$$\left\| \hat{\beta}_K' \left(\frac{\hat{\beta}_K - \beta_K^*}{\sqrt{n}} \right) \right\| = o_p(1).$$

(d) Suppose that $(n, T \to \infty)$ with $\frac{n}{T} \to 0$. Then,

$$\left\| \left(\frac{\hat{\beta}_K - \beta_K^*}{\sqrt{n}} \right)' \beta_K^* \right\| = o_p(1).$$

Proof of Lemma 8

Part (a).

Part (a) holds by modifying the proof of Lemma 9(a) of Moon and Phillips (2000) with $c = 0$ and $\tilde{h}_T(s,t) = 1$, and we omit the proof. ■

Part (b).

Denote $\sigma_{e,i}^2 = Ee_{it}^2$. Then, $\lambda_{e,i} = \frac{1}{2}\left(\omega_{e,i}^2 - \sigma_{e,i}^2\right)$ and

$$\frac{1}{T}\sum_{t=1}^{T} E_{it-1}e_{it} - \lambda_{e,i} = \frac{1}{2}\left(\frac{E_{iT}^2}{T} - \omega_{e,i}^2\right) - \frac{1}{2}\left(\frac{1}{T}\sum_{t=1}^{T}\left(e_{it}^2 - \sigma_{e,i}^2\right)\right).$$

Then, we have

$$\begin{aligned} & \sqrt{n}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E_{it-1}e_{it} - \lambda_e^n\right) \\ = & \frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\left(\frac{E_{iT}^2}{T} - \omega_{e,i}^2\right) - \frac{1}{2\sqrt{n}}\left(\frac{1}{T}\sum_{t=1}^{T}\left(e_{it}^2 - \sigma_{e,i}^2\right)\right) = I_b - II_b, \text{ say.} \end{aligned}$$

First, by Lemma 6(c) we have

$$E\left(II_b\right)^2 \to 0$$

as $(n, T \to \infty)$. Therefore

$$II_b = o_p(1).$$

Next, using the BN decomposition of e_{it}, we may write

$$\begin{aligned} & \frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\left(\frac{E_{iT}^2}{T} - \omega_{e,i}^2\right) \\ = & \frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\omega_{e,i}^2\left(\frac{V_{iT}^2}{T} - 1\right) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} d_i(1)\frac{V_{iT}}{\sqrt{T}}\frac{\left(\check{e}_{i0} - \check{e}_{iT}\right)}{\sqrt{T}} + \frac{1}{2\sqrt{n}T}\sum_{i=1}^{n}\left(\check{e}_{i0} - \check{e}_{iT}\right)^2 \\ = & I_{ba} + I_{bb} + I_{bc}, \text{ say,} \end{aligned}$$

where $V_{iT} = \sum_{s=1}^{T} v_{is}$. By Lemma 6(b), we have

$$E\left(I_{bc}\right) = \frac{\sqrt{n}}{2T}\frac{1}{n}\sum_{i=1}^{n} E\left(\check{e}_{i0} - \check{e}_{iT}\right)^2 \le \frac{2\sqrt{n}}{T}\sup_{i,t}\left(E\check{e}_{it}^2\right) \le \frac{\sqrt{n}}{T}M.$$

Since $I_{bc} \geq 0$ and $\frac{n}{T} \to 0$,

$$I_{bc} = o_p(1).$$

For I_{ba}, we apply Theorem 3 in Phillips and Moon (1999) and use Assumption 8. Then, we have

$$I_{ba} = \frac{1}{2\sqrt{n}} \sum_{i=1}^{n} \omega_{e,i}^2 \left(\frac{V_{iT}^2}{T} - 1 \right) \Rightarrow N\left(0, \frac{1}{2}\phi_e^4\right).$$

Finally, by the Cauchy-Schwarz inequality together with $I_{ba} = o_p(1)$ and $I_{bc} = O_p(1)$, we have

$$I_{bb} = o_p(1).$$

Therefore, as $(n, T \to \infty)$ with $\frac{n}{T} \to 0$,

$$\sqrt{n}\left(\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} E_{it-1} e_{it} - \lambda_e^n \right) = I_{ba} + o_p(1) \Rightarrow N\left(0, \frac{1}{2}\phi_e^4\right),$$

and we have the required result. ■

Proof of Lemma 9

Part (a).

Since

$$\begin{aligned} \frac{1}{nT^2} \|Z_{-1}\|^2 &\leq \frac{2}{nT^2} \|Z_{-1}(0)\|^2 + \frac{2}{nT^2} \|Z_{-1} - Z_{-1}(0)\|^2 \\ &= \frac{2}{nT^2} \|Z_{-1}(0)\|^2 + O_p\left(\frac{1}{n}\right) \end{aligned}$$

by Lemma 10(a), the required result follows if we show that

$$\frac{1}{nT^2} \|Z_{-1}(0)\|^2 = O_p(1).$$

By definition,

$$\frac{1}{nT^2} \|Z_{-1}(0)\|^2 = \frac{1}{nT^2} \left\| l_T \alpha' + F_{-1}^0 \beta^{0\prime} + E_{-1} \right\|^2 \leq 2\left(I_a + II_a + III_a\right),$$

where $I_a = \frac{1}{nT} \|z_0\|^2$, $II_a = \frac{1}{nT^2} \left\| F_{-1}^0 \beta^{0\prime} \right\|^2$, $III_a = \frac{1}{nT^2} \|E_{-1}\|^2$, $F_{-1}^0 = \Xi f_{-1}^0$, and $E_{-1} = \Xi e_{-1}$.

For I_a, since $\sup_i E\alpha_i^2 < M$ by Assumption 9,

$$I_a = O_p\left(\frac{1}{T}\right).$$

Next, for II_a, notice that

$$\begin{aligned} II_a &= \frac{1}{nT^2} \left\| F_{-1}^0 \beta^{0\prime} \right\|^2 = \frac{1}{nT^2} tr\left(\beta^{0\prime} \beta^0 F_{-1}^{0\prime} F_{-1}^0 \right) \\ &= tr\left(\left(\frac{1}{n} \sum_{i=1}^{n} \beta_i^0 \beta_i^{0\prime} \right) \left(\frac{1}{T^2} \sum_{t=1}^{T} F_{t-1}^0 F_{t-1}^{0\prime} \right) \right) = O_p(1), \end{aligned}$$

where the last equality holds because $\frac{1}{T^2}\sum_{t=1}^{T}F_{t-1}^{0}F_{t-1}^{0\prime}=O_p(1)$ under Assumption 3 and $\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}=O(1)$ under Assumption 6.

Finally, for III_a, as in the proof of Lemma 8(a) with $\theta_i=1$,

$$III_a=\frac{1}{nT^2}\left\|E_{-1}\right\|^2=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}E_{it-1}^2\rightarrow_p\frac{\omega_e^2}{2}.$$

Thus, $\frac{1}{nT^2}\left\|Z_{-1}(0)\right\|^2=O_p(1)$, as required. ■

Part (b).

Let Z_t' and y_t' denote the t^{th} rows of Z and y, respectively. Since

$$Z_{-1}=l_T\alpha'+Z_{-1}^0,$$

we have

$$\frac{1}{nT}\left\|Z_{-1}'y+y'Z_{-1}\right\|\leq\frac{2}{nT}\left\|y'l_T\alpha'\right\|+\frac{1}{nT}\left\|Z_{-1}^{0\prime}y+y'Z_{-1}^0\right\|.$$

First,

$$\begin{aligned}\frac{1}{nT}\left\|y'l_T\alpha'\right\| &\leq \frac{1}{\sqrt{T}}\frac{\|\alpha\|}{\sqrt{n}}\left\{\frac{\left\|\beta^0\right\|}{\sqrt{n}}\frac{\left\|f^{0\prime}l_T\right\|}{\sqrt{T}}+\frac{\left\|e'l_T\right\|}{\sqrt{nT}}\right\}\\ &= \frac{1}{\sqrt{T}}O_p(1)\left(O_p(1)+O_p(1)\right)=O_p\left(\frac{1}{\sqrt{T}}\right),\end{aligned}$$

where the first equality holds by Assumption 9, and Part (i*).

Next, using $Z_t^0=\rho Z_{t-1}^0+y_t$, we may write

$$\begin{aligned}Z_{-1}^{0\prime}y+y'Z_{-1}^0 &= \sum_{t=1}^{T}\left(Z_{t-1}^0y_t'+y_tZ_{t-1}^{0\prime}\right)\\ &= Z_T^0Z_T^{0\prime}+\rho Z_{-1}^{0\prime}Z_{-1}^0\rho-Z_{-1}^{0\prime}Z_{-1}^0+\left(\rho-I_n\right)Z_{-1}^{0\prime}y+y'Z_{-1}^0\left(\rho-I_n\right)+y'y.\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{nT}\left\|Z_{-1}'y+y'Z_{-1}\right\|\\ \leq\ &\frac{\left\|Z_T^0\right\|^2}{nT}+\frac{1}{T}\left(T\left\|\rho-I_n\right\|\right)^2\frac{\left\|Z_{-1}^0\right\|^2}{nT^2}+2\left(T\left\|\rho-I_n\right\|\right)\frac{\left\|Z_{-1}^0\right\|^2}{nT^2}\\ &+\frac{2}{\sqrt{T}}\left(T\left\|\rho-I_n\right\|\right)\frac{\left\|Z_{-1}^0\right\|}{\sqrt{n}T}\frac{\|y\|}{\sqrt{nT}}+\frac{\|y\|^2}{nT}.\end{aligned}\tag{20}$$

By Part (a), $\frac{\left\|Z_{-1}^0\right\|^2}{nT^2}=O_p(1)$. Similarly we can show that $\frac{\left\|Z_T^0\right\|^2}{nT}=O_p(1)$. Notice that $T\left\|\rho-I_n\right\|=\frac{1}{\sqrt{n}}\|\Theta\|=O_p(1)$ and $\frac{\|y\|^2}{nT}=O_p(1)$ by Part (k). Therefore,

$$(20)=O_p(1),$$

as required. ■

Part (c).

Notice that

$$\begin{aligned}
\frac{1}{nT^2}tr\left(\beta^{0\prime}E_{-1}^{\prime}E_{-1}\beta^{0}\right) &= tr\left(\frac{1}{T^2}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 E_{it-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 E_{it-1}\right)^{\prime}\right) \\
&= \frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 E_{it-1}\right\|^2. \qquad (21)
\end{aligned}$$

Define $V_{it}=\sum_{s=1}^{t}v_{is}$ with $V_{i0}=0$. Using the BN decomposition of e_{it}, we have

$$E_{it-1}=d_iV_{it-1}+\breve{e}_{i0}-\breve{e}_{it-1},$$

where $d_i=d_i\left(1\right)$. Plugging this into (21),

$$\begin{aligned}
(21) &= \frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\left(d_iV_{it-1}+\breve{e}_{i0}-\breve{e}_{it-1}\right)\right\|^2 \\
&\leq 2\frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 d_iV_{it-1}\right\|^2+2\frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\left(\breve{e}_{i0}-\breve{e}_{it-1}\right)\right\|^2 \\
&= 2\left(I_c+II_c\right),\ \text{say.}
\end{aligned}$$

Set $\bar{d}=\sum_{j=0}^{\infty}\bar{d}_j$. Since $I_c\geq 0$ and

$$\begin{aligned}
EI_c &= E\left(\frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 d_iV_{it-1}\right\|^2\right)=tr\left[\frac{1}{T^2}\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 d_iV_{it-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0 d_iV_{it-1}\right)^{\prime}\right] \\
&= \left(\frac{1}{T^2}\sum_{t=1}^{T}(t-1)\right)tr\left(\frac{1}{n}\sum_{i=1}^{n}d_i^2\beta_i^0\beta_i^{0\prime}\right)\leq\bar{d}^2\left(\frac{1}{T^2}\sum_{t=1}^{T}(t-1)\right)tr\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\right)=O\left(1\right),
\end{aligned}$$

we have

$$I_c=O_p\left(1\right).$$

Similarly, since $II_c\geq 0$ and

$$\begin{aligned}
EII_c &= E\left(\frac{1}{T^2}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\left(\breve{e}_{i0}-\breve{e}_{it-1}\right)\right\|^2\right) \\
&= tr\left[\frac{1}{T^2}\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\left(\breve{e}_{i0}-\breve{e}_{it-1}\right)\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\left(\breve{e}_{i0}-\breve{e}_{it-1}\right)\right)^{\prime}\right] \\
&\leq \frac{M}{T}tr\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\right)=o\left(1\right),
\end{aligned}$$

where the inequality holds by Lemma 6(b), we have

$$II_c=o_p\left(1\right),$$

which completes the proof of Part (c). ■

Part (d).

Notice that

$$
\begin{aligned}
& \frac{1}{nT}\left\|\beta^{0\prime}e'E_{-1}\beta^{0}+\beta^{0\prime}E_{-1}'e\beta^{0}\right\| = \frac{1}{nT}\left\|\beta^{0\prime}\left(E_{T-1}E_{T-1}'-\sum_{t=1}^{T}e_te_t'\right)\beta^{0}\right\| \\
\leq\ & \frac{1}{nT}\left\|\beta^{0\prime}E_{T-1}E_{T-1}'\beta^{0}\right\| + \frac{1}{nT}\left\|\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\beta_i^{0}e_{it}\right)\left(\sum_{i=1}^{n}\beta_i^{0}e_{it}\right)'\right\| \\
\leq\ & \frac{1}{nT}\left\|\beta^{0\prime}E_{T-1}E_{T-1}'\beta^{0}\right\| + \frac{1}{T}\sum_{t=1}^{T}\left\|\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^{0}e_{it}\right)\right\|^{2} \\
=\ & I_d + II_d,\ \text{say.}
\end{aligned}
$$

Using arguments similar to those in the proof of Part (c), we can show that

$$I_d = O_p(1).$$

For II_d, note that

$$EII_d = \frac{1}{T}\sum_{t=1}^{T} tr\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0}\beta_i^{0\prime}Ee_{it}^{2}\right).$$

Since $\sup_{i,t} Ee_{it}^{2} < M$ (see Lemma 6(a)) and $\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0}\beta_i^{0\prime} \to \Sigma_\beta$ by Assumption 6,

$$EII_d \leq M\left[tr\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0}\beta_i^{0\prime}\right)\right] = O(1).$$

Since $II_d \geq 0$,

$$II_d = O_p(1), \tag{22}$$

and we have all the required results. ■

Part (e).

Denote $\Gamma_{e,i}(h) = E(e_{it}e_{it-h})$. Let $e_t = (e_{1t},...,e_{nt})'$. Notice that

$$
\begin{aligned}
E\left\|\frac{e'e}{nT}\right\|^{2} =\ & E\left\|\frac{\sum_{t=1}^{T}e_te_t'}{nT}\right\|^{2} = E\left(\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}e_t'e_se_s'e_t\right) \\
=\ & \frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left(\sum_{i=1}^{n}e_{it}e_{is}\right)\left(\sum_{j=1}^{n}e_{jt}e_{js}\right) \\
=\ & \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\frac{1}{n^2}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\Gamma_{e,i}(t-s)\Gamma_{e,j}(t-s)\right) + \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{1}{n^2}\sum_{i=1}^{n}Ee_{it}^{2}e_{is}^{2} \\
\leq\ & \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\frac{1}{n}\sum_{i=1}^{n}\Gamma_{e,i}(t-s)\right)^{2} + \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\frac{1}{n^2}\sum_{i=1}^{n}\sqrt{E\left(e_{it}^{4}\right)E\left(e_{is}^{4}\right)} \\
=\ & I_e + II_e,\ \text{say,}
\end{aligned}
$$

where the inequality holds by the Cauchy-Schwarz inequality.

Under Assumption 2,

$$\sup_i |\Gamma_{e,i}(h)| \leq \sup_i \sum_{j=0}^{\infty} |d_{ij} d_{ij+h}| \leq \sum_{j=0}^{\infty} \bar{d}_j \bar{d}_{j+h} \overset{let}{=} \bar{\Gamma}_e(h). \tag{23}$$

So,

$$I_e \leq \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \bar{\Gamma}_e(t-s)^2 \leq \frac{2}{T} \sum_{h=0}^{\infty} \bar{\Gamma}_e(h)^2. \tag{24}$$

Since

$$\sum_{h=0}^{\infty} \bar{\Gamma}_e(h)^2 = \sum_{h=0}^{\infty} \left(\sum_{j=0}^{\infty} \bar{d}_j \bar{d}_{j+h} \right)^2 \leq \left(\sum_{h=0}^{\infty} \sum_{j=0}^{\infty} \bar{d}_j \bar{d}_{j+h} \right)^2 \leq \left(\sum_{j=0}^{\infty} \bar{d}_j \right)^4 < M, \tag{25}$$

we have

$$\text{RHS of } (24) = O\left(\frac{1}{T}\right),$$

and so,

$$I_e = O\left(\frac{1}{T}\right).$$

Next, since

$$\sup_{i,t} E e_{it}^4 < M$$

by Lemma 6(a), we have

$$II_e = \frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{T} \frac{1}{n^2} \sum_{i=1}^{n} \sqrt{E(e_{it}^4) E(e_{is}^4)} = O\left(\frac{1}{n}\right).$$

In consequence,

$$E \left\| \frac{e'e}{nT} \right\|^2 = O\left(\frac{1}{T}\right) + O\left(\frac{1}{n}\right)$$

and so,

$$\left\| \frac{e'e}{nT} \right\| = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{n}} \right) \right),$$

as required. ■

Part (f).

Notice that

$$\begin{aligned}
\frac{1}{nT} tr\left(f^{0\prime} e e' f^0\right) &= \frac{1}{nT} tr\left(\sum_{t=1}^{T} f_t^0 e_t' \right) \left(\sum_{s=1}^{T} e_s f_s^{0\prime} \right) = tr\left(\frac{1}{nT} \sum_{t=1}^{T} \sum_{s=1}^{T} f_t^0 f_s^{0\prime} \left(\sum_{i=1}^{n} e_{it} e_{is} \right) \right) \\
&= \frac{1}{n} \sum_{i=1}^{n} tr\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_t^0 e_{it} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_t^0 e_{it} \right)' = \frac{1}{n} \sum_{i=1}^{n} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_t^0 e_{it} \right\|^2.
\end{aligned}$$

Denote $\Gamma_f(h) = E\left(f_t^0 f_{t-h}^{0\prime}\right)$. Under Assumptions 2 and 3,

$$
\begin{aligned}
& E\left[\frac{1}{n}\sum_{i=1}^{n}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} f_t^0 e_{it}\right\|^2\right] = tr\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_f(t-s)\,\Gamma_{e,i}(t-s)\right] \\
\leq\ & \sqrt{K}\left\|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_f(t-s)\,\Gamma_{e,i}(t-s)\right\| \\
\leq\ & \frac{\sqrt{K}}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|\Gamma_f(t-s)\right\|\bar{\Gamma}_e(t-s) \leq \sqrt{K}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|\Gamma_f(t-s)\right\|^2}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\bar{\Gamma}_e(t-s)^2} \\
\leq\ & 2\sqrt{K}\sqrt{\sum_{h=0}^{\infty}\left\|\Gamma_f(h)\right\|^2}\sqrt{\sum_{h=0}^{\infty}\bar{\Gamma}_e(h)^2} < M,
\end{aligned}
$$

where the first equality holds because f_t^0 and e_{it} are assumed to be independent, the first inequality holds by the definition of $\bar{\Gamma}_e(t-s)$ in (23), the second inequality is the Cauchy-Schwarz inequality, and last inequality holds due to the summability conditions of Assumptions 2(ii) and 3(ii).

Since $\frac{1}{nT}tr\left(f^{0\prime}ee'f^0\right) \geq 0$ and $E\frac{1}{nT}tr\left(f^{0\prime}ee'f^0\right) < M$, we have the required result,

$$
\frac{1}{nT}tr\left(f^{0\prime}ee'f^0\right) = O_p(1). \blacksquare
$$

Part (g).

Letting $b_{ij} = \left(\beta_i^0\right)'\beta_j^0$, notice that

$$
\begin{aligned}
tr\left(\frac{\beta^{0\prime}e'e\beta^0}{nT}\right) &= \frac{1}{nT}tr\left(\sum_{j=1}^{n}\beta_j^0\underline{e}_j'\right)\left(\sum_{i=1}^{n}\underline{e}_i\beta_i^{0\prime}\right) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}\sum_{t=1}^{T}e_{it}e_{jt} \\
&= \frac{1}{nT}\sum_{i=1}^{n}b_{ii}\sum_{t=1}^{T}e_{it}^2 + \frac{1}{nT}\sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n}b_{ij}\sum_{t=1}^{T}e_{it}e_{jt} = I_g + II_g, \text{ say.}
\end{aligned}
$$

First, since $\sup_{it}Ee_{it}^2 < M$ by Lemma 6(a) and by Assumption 6,

$$
E\left|I_g\right| \leq M\left(tr\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\right)\right) = O(1),
$$

and so,

$$
I_g = O_p(1).
$$

Next, for II_g, consider $E\left(II_g\right)^2$. Since

$$\begin{aligned}
E\left(II_g\right)^2 &= \frac{1}{n^2T}E\left(\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}b_{ij}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}e_{jt}\right)\left(\sum_{\substack{k=1\\k\neq l}}^{n}\sum_{l=1}^{n}b_{kl}\frac{1}{\sqrt{T}}\sum_{s=1}^{T}e_{ks}e_{ls}\right)\\
&\leq M_1\frac{1}{n^2T}\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}b_{ij}^2E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}e_{jt}\right)^2 \text{ by the independence of } e_{it} \text{ across } i\\
&\leq M_2\frac{1}{n^2T}\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}b_{ij}^2\left(\sum_{h=0}^{T}\left(1-\frac{h}{T}\right)\Gamma_{e,i}\left(h\right)\Gamma_{e,j}\left(h\right)\right)\\
&\leq \frac{M_2}{T}\left(\frac{1}{n^2}\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}b_{ij}^2\right)\left(\sum_{h=0}^{\infty}\bar{\Gamma}_e\left(h\right)^2\right)\rightarrow 0,
\end{aligned}$$

where M_1 and M_2 are finite constants and the last convergence holds because $\left(\sum_{h=0}^{\infty}\bar{\Gamma}_e\left(h\right)^2\right) < M$ by (25) and $\frac{1}{n^2}\sum_{\substack{i=1\\i\neq j}}^{n}\sum_{j=1}^{n}b_{ij}^2 = O\left(1\right)$ by Assumption 6. Therefore, we have

$$II_g = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p\left(1\right),$$

and we have all the required result that

$$tr\left(\frac{\beta^{0\prime}e^{\prime}e\beta^0}{nT}\right) = O_p\left(1\right). \blacksquare$$

Part (h).

We can write

$$\begin{aligned}
&\frac{1}{n\sqrt{n}T\sqrt{T}}\left\|\beta^{0\prime}e^{\prime}eE_T\right\|\\
=\ &\frac{1}{n\sqrt{n}T\sqrt{T}}\left\|\sum_{i=1}^{n}\beta_i^0\underline{e}_i^{\prime}\sum_{j=1}^{n}\underline{e}_jE_{jT}\right\| = \frac{1}{n\sqrt{n}T\sqrt{T}}\left\|\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_i^0\sum_{t=1}^{T}e_{it}e_{jt}\left(\sum_{s=1}^{T}e_{js}\right)\right\|\\
\leq\ &\frac{\sqrt{n}}{\sqrt{T}}\left\|\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_i^0X_{ij,T}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}e_{js}\right)\right\| + \frac{1}{n}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0E\left(e_{it}^2\right)\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}e_{is}\right)\right\|\\
=\ &I_h + II_h,\ \text{say},
\end{aligned}$$

where $X_{ij,T} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(e_{it}e_{jt} - E\left(e_{it}e_{jt}\right)\right)$. First, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
I_h &= \frac{\sqrt{n}}{\sqrt{T}}\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|\beta_i^0\right\|^2X_{ij,T}^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right)^2}\\
&= \frac{\sqrt{n}}{\sqrt{T}}O_p\left(1\right)O_p\left(1\right) = o_p\left(1\right),
\end{aligned}$$

where the last equality holds because $\sup_{i,j} EX^2_{ij,T} < M$ by Lemma 7, $\frac{1}{n}\sum_{i=1}^n \left\|\beta_i^0\right\|^2 = O(1)$ by Assumption 6, and $\sup_{i,T} E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T e_{it}\right)^2 < M$.

Next, for some finite constant M_1,

$$EII_h^2 \le \frac{1}{n^2}E\left(\frac{1}{n}\sum_{i=1}^n tr\left(\beta_i^0\beta_i^{0\prime}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T e_{it}\right)^2\left(E\left(e_{it}^2\right)\right)^2\right) = O\left(\frac{1}{n^2}\right),$$

where the last equality holds because $\sup_i E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^T e_{it}\right)^2 < M$, $\sup_{i,t} E\left(e_{it}^2\right) < M$, $\frac{1}{n}\sum_{i=1}^n tr\left(\beta_i^0\beta_i^{0\prime}\right) = O(1)$ by Assumption 6. Thus,

$$II_h = O_p\left(\frac{1}{n}\right).$$

Together, we have the required result,

$$\frac{1}{n\sqrt{n}T\sqrt{T}}\left\|\beta^{0\prime}e'eE_T\right\| = O_p\left(\frac{\sqrt{n}}{\sqrt{T}}\right) + O_p\left(\frac{1}{n}\right) = o_p(1). \blacksquare$$

Part (i).

Since

$$\begin{aligned}E\left[\frac{1}{nT}\left\|\beta^{0\prime}e'l_T\right\|^2\right] &= \frac{1}{nT}E\left\|\sum_{i=1}^n\sum_{t=1}^T \beta_i^0 e_{it}\right\|^2 = \frac{1}{n}\sum_{i=1}^n tr\left(\beta_i^0\beta_i^{0\prime}\right)\frac{1}{T}\sum_{t=1}^T\sum_{s=1}^T \Gamma_{e,i}(t-s)\\ &\le 2\left(\frac{1}{n}\sum_{i=1}^n tr\left(\beta_i^0\beta_i^{0\prime}\right)\right)\left(\sum_{h=0}^\infty \bar{\Gamma}_e(h)\right) < M,\end{aligned}$$

we have the required result,

$$\frac{1}{\sqrt{nT}}\left\|\beta^{0\prime}e'l_T\right\| = O_p(1). \blacksquare$$

Part (i*).

Since

$$\begin{aligned}E\left[\frac{1}{nT}\left\|e'l_T\right\|^2\right] &= \frac{1}{nT}\sum_{t=1}^T\sum_{s=1}^T\sum_{i=1}^n E\left(e_{it}e_{is}\right) = \frac{1}{T}\sum_{t=1}^T\sum_{s=1}^T\frac{1}{n}\sum_{i=1}^n \Gamma_{e,i}(t-s)\\ &\le \frac{1}{T}\sum_{t=1}^T\sum_{s=1}^T \bar{\Gamma}_e(t-s) \le 2\sum_{h=0}^\infty \bar{\Gamma}_e(h) < M,\end{aligned}$$

and the required result follows. $\blacksquare$

Part (i**).

Part (i**) follows because

$$\frac{\left\|y'l_T\right\|}{\sqrt{nT}} \le \frac{\left\|\beta^0\right\|}{\sqrt{n}}\frac{\left\|f^{0\prime}l_T\right\|}{\sqrt{T}} + \frac{\left\|e'l_T\right\|}{\sqrt{nT}} = O_p(1) + O_p(1) = O_p(1). \blacksquare$$

Part (j).

The required result follows because

$$\frac{1}{\sqrt{n}T\sqrt{T}}\left\|\beta^{0\prime}E_{-1}^{\prime}l_T\right\| = \left\|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}\sqrt{T}}\sum_{i=1}^{n}\beta^0 E_{it-1}\right)\right\| \leq \frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}\sqrt{T}}\sum_{i=1}^{n}\beta^0 E_{it-1}\right\|^2 = O_p(1),$$

where the last equality holds by Part (c). ■

Part (k)

Part (k) follows since

$$\begin{aligned}
\frac{1}{nT}\|y\|^2 &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}y_{it}^2 \\
&\leq 2\left[tr\left(\frac{1}{T}\sum_{t=1}^{T}f_t^0 f_t^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\right) + \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^2\right] \\
&= O_p(1) + O_p(1) = O_p(1),
\end{aligned}$$

where the last line holds by Assumptions 7 and 6 and Part (e).

Part (l)

Notice that by the Cauchy-Schwarz inequality

$$\frac{1}{nT\sqrt{T}}\left|\alpha^{\prime}E_{-1}^{\prime}l_T\right| = \left|\frac{1}{n}\sum_{i=1}^{n}\alpha_i\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}E_{it-1}\right| \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}E_{it-1}\right)^2}.$$

By Assumption **??**, $\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2 = O_p(1)$ since $\sup_i E\alpha_i^2 < M$. Also, it is possible to show that $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}E_{it-1}\right)^2 = O_p(1)$ (*c.f.*, see Lemma 9 of Moon and Phillips, 2000). Therefore, we have the required result,

$$\frac{1}{nT\sqrt{T}}\left|\alpha^{\prime}E_{-1}^{\prime}l_T\right| = O_p(1). \blacksquare$$

Proof of Lemma 10

Parts (a) and (b). Parts (a) and (b) follows because

$$\begin{aligned}
&\frac{1}{nT^2}E\left(\sum_{i=1}^{n}\theta_i^2\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\frac{(t-s-1)}{T}\beta_i^{0\prime}f_s^0\right)^2\right) \\
\leq\ &\bar{M}_\theta^2\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{q=1}^{t-1}\frac{(t-s-1)(t-q-1)}{T^2}\left(\beta_i^{0\prime}\Gamma_f(s-q)\beta_i^0\right)\right) \\
\leq\ &\bar{M}_\theta^2\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\beta_i^0\right\|^2\right)\left(\frac{1}{T}\sum_{s=1}^{T}\sum_{q=1}^{T}\left\|\Gamma_f(s-q)\right\|\right) < \infty,
\end{aligned}$$

and

$$\frac{1}{nT^2}E\left(\sum_{i=1}^{n}\theta_i^2\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\frac{(t-s-1)}{T}e_{is}\right)^2\right)\leq\bar{M}_\theta^2\left(\frac{1}{T}\sum_{s=1}^{T}\sum_{q=1}^{T}\bar{\Gamma}_e\left(s-q\right)\right)<\infty.\ \blacksquare$$

Parts (c), (d), and (e). Parts (c),(d), and (e) follow because

$$\begin{aligned}
& E\left(\frac{1}{nT}\sum_{i=1}^{n}\theta_i\beta_i^{0\prime}\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)f_s^0f_t^{0\prime}\right)\beta_i^0\right)^2 \\
= & \frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(\theta_i\theta_j\right)\beta_i^{0\prime}E\left[\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)f_s^0f_t^{0\prime}\right)\beta_i^0\beta_j^{0\prime}\left(\sum_{p=2}^{T}\sum_{q=1}^{p-1}\left(\frac{p-q-1}{T}\right)f_p^0f_q^{0\prime}\right)\right]\beta_j^0 \\
= & O\left(1\right),
\end{aligned}$$

$$\begin{aligned}
& E\left(\frac{1}{nT}\sum_{i=1}^{n}\theta_i\beta_i^{0\prime}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)f_s^0e_{it}\right)^2 \\
= & \frac{1}{n^2T^2}\sum_{i=1}^{n}E\left(\theta_i^2\right)\beta_i^{0\prime}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}\left(\frac{t-s-1}{T}\right)\left(\frac{p-q-1}{T}\right)E\left(f_s^0f_q^{0\prime}\right)E\left(e_{it}e_{ip}\right)\beta_i^0 \\
\leq & \frac{1}{n^2T^2}\sum_{i=1}^{n}E\left(\theta_i^2\right)\beta_i^{0\prime}\beta_i^0\left(\sum_{t=1}^{T}\sum_{p=1}^{T}\bar{\Gamma}_e\left(t-p\right)\right)\left(\sum_{s=1}^{T}\sum_{q=1}^{T}\left\|\Gamma_f\left(s-q\right)\right\|\right)=O\left(\frac{1}{n}\right),
\end{aligned}$$

and

$$\begin{aligned}
& E\left(\frac{1}{nT}\sum_{i=1}^{n}\theta_i\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)e_{is}e_{it}\right)^2 \\
= & \frac{1}{n^2T^4}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(\theta_i\theta_j\right)\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}\left(t-s-1\right)\left(p-q-1\right)E\left(e_{is}e_{it}e_{jq}e_{jp}\right) \\
\leq & \bar{M}_\theta^2\left[\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{p=1}^{T}\sum_{q=1}^{T}\left|E\left(e_{is}e_{it}e_{iq}e_{ip}\right)\right|+\left(\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(t-s-1\right)\bar{\Gamma}_e\left(t-s\right)\right)^2\right] \\
= & O\left(\frac{1}{n}\right)+\mu_\theta^2\left(\sum_{h=1}^{T-1}\left(1-\frac{h}{T}\right)\frac{h}{T}\bar{\Gamma}_e\left(h\right)\right)^2=o\left(1\right).\ \blacksquare
\end{aligned}$$

Parts (f) and (g). Parts (f) and (g) follow because

$$
\begin{aligned}
& E\left\|\frac{1}{n^2T}\sum_{i=1}^{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\theta_j\sum_{t=2}^{T}e_{it}\left(\sum_{s=1}^{t-1}\frac{t-s-1}{T}f_s^0\right)\beta_j^{0\prime}\right\|^2 \\
= & \frac{1}{n^4T^4}tr\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\theta_j\theta_l\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}(t-s-1)(p-q-1)E(e_{it}e_{kp})\beta_j^{0\prime}E\left(f_s^0f_q^{0\prime}\right)\beta_l^0\beta_j^{0\prime}\beta_l^0\beta_k^{0\prime}\beta_i^0\right) \\
= & \frac{1}{n^4T^4}tr\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_j\theta_l\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}(t-s-1)(p-q-1)\beta_i^{0\prime}\beta_i^0E(e_{it}e_{ip})\beta_j^{0\prime}E\left(f_s^0f_q^{0\prime}\right)\beta_l^0\beta_j^{0\prime}\beta_l^0\right) \\
\leq & \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\beta_i^0\right\|^2\right)\left(\frac{1}{n}\sum_{j=1}^{n}\theta_j\left\|\beta_j^0\right\|^2\right)^2\left(\frac{1}{T}\sum_{t=2}^{T}\sum_{p=2}^{T}\sup_i\left|E(e_{it}e_{ip})\right|\right)\left(\frac{1}{T}\sum_{s=1}^{T}\sum_{q=1}^{T}\left\|E\left(f_s^0f_q^{0\prime}\right)\right\|\right) \\
= & \frac{1}{n}O(1)O_p(1)O(1)O(1)=o(1),
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left(\left\|\frac{1}{n^2T}\sum_{i=1}^{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\theta_j\sum_{t=2}^{T}e_{it}\left(\sum_{s=1}^{t-1}\frac{t-s-1}{T}e_{js}\right)\beta_j^{0\prime}\right\|^2\right) \\
= & \frac{1}{n^4T^4}trE\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\beta_i^0\beta_j^{0\prime}\beta_l^0\beta_k^{0\prime}\theta_j\theta_l\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}(t-s-1)(p-q-1)e_{it}e_{js}e_{kp}e_{lq}\right) \\
= & O\left(\frac{1}{n^2}\right)=o(1). \blacksquare
\end{aligned}
$$

Parts (h) and (i) Parts (h) and (i) follow because

$$
\frac{1}{n^2T}E\left\|\sum_{i=1}^{n}\theta_i\beta_i^0\beta_i^{0\prime}\left(\sum_{s=1}^{T}\left(1-\frac{s}{T}\right)f_s^0\right)\right\|^2\leq\bar{M}_\theta^2\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\beta_i^0\right\|^2\right)^2E\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(1-\frac{t}{T}\right)f_t^0\right\|^2=O(1),
$$

and

$$
\begin{aligned}
& E\left(\frac{1}{n\sqrt{T}}\left\|\sum_{i=1}^{n}\beta_i^0\theta_i\left(\sum_{s=1}^{T}\left(1-\frac{s}{T}\right)e_{is}\right)\right\|\right)^2 \\
\leq & \frac{\bar{M}_\theta}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\beta_i^0\right\|^2\right)\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(1-\frac{t}{T}\right)\left(1-\frac{s}{T}\right)\bar{\Gamma}_e(t-s)\right)=O\left(\frac{1}{n}\right)=o(1). \blacksquare
\end{aligned}
$$

Proof of Lemma 11

See Moon and Perron (2002). ■

Proof of Lemma 12

See Moon and Perron (2002). ■

2.2 Appendix B: Proofs of the Results in Section 2.1

Proof of Lemma 1.

By definition,

$$T\left(\hat{\rho}_{pool}-1\right)=\frac{\frac{1}{nT}tr\left(Z_{-1}'\left(Z-Z_{-1}\right)\right)}{\frac{1}{nT^2}tr\left(Z_{-1}'Z_{-1}\right)}.$$

We show

$$(a)\ \frac{1}{nT^2}tr\left(Z_{-1}'Z_{-1}\right)\Rightarrow tr\left(\int_0^1 B_f\left(r\right)B_f\left(r\right)'dr\right)\Sigma_\beta+\frac{1}{2}\omega_e^2$$

and

$$(b)\ \frac{1}{nT}tr\left(Z_{-1}'\left(Z-Z_{-1}\right)\right)\Rightarrow\frac{1}{2}tr\left(B_f\left(1\right)B_f\left(1\right)'\Sigma_\beta\right)+\frac{1}{2}\omega_e^2-\frac{1}{2}tr\left(\Sigma_f\Sigma_\beta\right)-\frac{1}{2}\sigma_e^2,$$

then the required result follows.

Part (a)

For (a), we show that

$$(a_1)\ \frac{1}{nT^2}tr\left(Z_{-1}'Z_{-1}-Z_{-1}'\left(0\right)Z_{-1}\left(0\right)\right)=o_p\left(1\right)$$

and then

$$(a_2)\ \frac{1}{nT^2}tr\left(Z_{-1}'\left(0\right)Z_{-1}\left(0\right)\right)\Rightarrow tr\left(\int_0^1 B_f\left(r\right)B_f\left(r\right)'dr\right)\Sigma_\beta+\frac{1}{2}\omega_e^2.$$

First, for Part (a$_1$), notice that

$$\begin{aligned}&\left|\frac{1}{nT^2}tr\left(Z_{-1}'Z_{-1}-Z_{-1}'\left(0\right)Z_{-1}\left(0\right)\right)\right|\\ \leq\ &\frac{1}{nT^2}\left|tr\left\{\left(Z_{-1}'-Z_{-1}'\left(0\right)\right)Z_{-1}\right)\right\}\right|+\frac{1}{nT^2}\left|tr\left\{Z_{-1}\left(0\right)'\left(Z_{-1}-Z_{-1}\left(0\right)\right)\right\}\right|\\ \leq\ &\frac{1}{\sqrt{n}}\frac{\left\|Z_{-1}-Z_{-1}\left(0\right)\right\|}{T}\frac{\left(\left\|Z_{-1}\right\|+\left\|Z_{-1}\left(0\right)\right\|\right)}{\sqrt{n}T}=O_p\left(\frac{1}{\sqrt{n}}\right)=o_p\left(1\right),\end{aligned}$$

where the first equality holds by Lemma 9(a) and Lemma 10(a).

Next, for Part (a$_2$) notice that

$$\begin{aligned}&\frac{1}{nT^2}tr\left(Z_{-1}'\left(0\right)Z_{-1}\left(0\right)\right)\\ =\ &\frac{1}{nT^2}tr\left(\alpha l_T'+\beta^0F_{-1}^{0\prime}+E_{-1}'\right)\left(l_T\alpha'+F_{-1}^0\beta^{0\prime}+E_{-1}\right)\\ =\ &\frac{1}{nT}\alpha'\alpha+\frac{2}{nT^2}l_T'\left(F_{-1}^0\beta^{0\prime}+E_{-1}\right)\alpha+\frac{1}{nT^2}tr\left(\beta^0F_{-1}^{0\prime}+E_{-1}'\right)\left(F_{-1}^0\beta^{0\prime}+E_{-1}\right)\\ =\ &I_a+2II_a+III_a,\ \text{say.}\end{aligned}$$

By Assumption 9,

$$I_a=\frac{1}{nT}\sum_{i=1}^n\alpha_i^2=O_p\left(\frac{1}{T}\right)=o_p\left(1\right).$$

Next,

$$
\begin{aligned}
|II_a| &\leq \left|\frac{\left(l_T' F_{-1}^0\right)\left(\beta^{0\prime}\alpha\right)}{nT^2}\right| + \left|\frac{l_T' E_{-1}\alpha}{nT^2}\right| \\
&\leq \frac{1}{\sqrt{T}}\left\|\frac{1}{T\sqrt{T}}\sum_{t=1}^{T} F_{t-1}\right\|\left\|\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\alpha_i\right\| + \frac{1}{\sqrt{T}}\left|\frac{l_T' E_{-1}\alpha}{nT\sqrt{T}}\right|.
\end{aligned}
$$

Under Assumption 3, $\frac{1}{T\sqrt{T}}\sum_{t=1}^{T} F_{t-1} = O_p(1)$. Under Assumptions 6 and 9, $\left\|\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\alpha_i\right\| \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left\|\beta_i^0\right\|^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\alpha_i^2} = O(1)\,O_p(1) = O_p(1)$. Also, by Lemma 9(l), $\frac{l_T' E_{-1}\alpha}{nT\sqrt{T}} = O_p(1)$. Therefore,

$$
II_a = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1).
$$

Finally,

$$
\begin{aligned}
III_a &= \frac{1}{nT^2} tr\left(\beta^0 F_{-1}^{0\prime} + E_{-1}'\right)\left(F_{-1}^0\beta^{0\prime} + E_{-1}\right) \\
&= tr\left(\left(\frac{F_{-1}^{0\prime}F_{-1}^0}{T^2}\right)\left(\frac{\beta^{0\prime}\beta^0}{n}\right)\right) + 2\frac{1}{nT^2} tr\left(\beta^0 F_{-1}^{0\prime}E_{-1}\right) + \frac{1}{nT^2} tr\left(E_{-1}'E_{-1}\right) \\
&= III_{aa} + 2III_{ab} + III_{ac}, \text{ say.}
\end{aligned}
$$

We start with III_{ab}. Notice that since $tr(AB) \leq \|A\|\,\|B\|$ and using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
III_{ab} &= \frac{1}{nT^2} tr\left(\beta^0 F_{-1}^{0\prime}E_{-1}\right) = tr\left(\frac{1}{nT^2}\sum_{t=1}^{T} F_{t-1}\left(\sum_{i=1}^{n}\beta_i^{0\prime}E_{it-1}\right)\right) \\
&\leq \frac{1}{\sqrt{n}}\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{F_{t-1}}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\frac{E_{it-1}}{\sqrt{T}}\right\| \\
&\leq \frac{1}{\sqrt{n}}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{F_{t-1}}{\sqrt{T}}\right\|^2}\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\frac{E_{it-1}}{\sqrt{T}}\right\|^2}.
\end{aligned}
$$

Under Assumption 3, $\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{F_{t-1}}{\sqrt{T}}\right\|^2 = \int_0^1\|B_f(r)\|^2 dr = O_p(1)$. By Lemma 9(c), $\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^0\frac{E_{it-1}}{\sqrt{T}}\right\|^2 = O_p(1)$. Therefore,

$$
III_{ab} = O_p\left(\frac{1}{\sqrt{n}}\right) = o_p(1).
$$

Next we consider the term III_{aa}. Notice that under Assumption 3, we have

$$
\frac{1}{T^2}\sum_{t=1}^{T} F_{t-1}F_{t-1}' \Rightarrow \int_0^1 B_f(r)\,B_f(r)'\,dr.
$$

So, in view of Assumption 6 and the continuous mapping theorem, we have the required result,

$$
III_{aa} = tr\left(\left(\frac{F_{-1}^{0\prime}F_{-1}^0}{T^2}\right)\left(\frac{\beta^{0\prime}\beta^0}{n}\right)\right) \Rightarrow tr\left(\int_0^1 B_f(r)\,B_f(r)'\,dr\right)\Sigma_\beta.
$$

For III_{ac}, by Lemma 8(a), we have

$$III_{ac} = \frac{1}{nT^2} \sum_{i=1}^{n} \sum_{t=1}^{T} E_{it-1}^2 \to_p \frac{1}{2}\omega_e^2.$$

Therefore, we have the required result for Part (a$_2$) as $(n, T \to \infty)$,

$$\frac{1}{nT^2} tr\left(Z'_{-1}(0)\, Z_{-1}(0)\right) \Rightarrow tr\left(\int_0^1 B_f(r)\, B_f(r)'\, dr\right) \Sigma_\beta + \frac{1}{2}\omega_e^2.$$

Combining Part (a$_1$) and (a$_2$), we have the desired result for Part (a).

Part (b)

Notice that

$$\begin{aligned}
\frac{1}{nT} tr\left(Z'_{-1}(Z - Z_{-1})\right) &= \frac{1}{nT} tr\left(Z'_{-1}\left(-\frac{1}{\sqrt{nT}} Z^0_{-1}\Theta + y\right)\right) \\
&= -\frac{1}{n\sqrt{n}T^2} tr\left(Z'_{-1} Z^0_{-1}\Theta\right) + \frac{1}{nT} tr\left(Z'_{-1} y\right) \\
&= -I_b + II_b, \text{ say.}
\end{aligned}$$

For part I_b, by definition

$$I_b = \frac{1}{n\sqrt{n}T^2} tr\left(\alpha l'_T Z^0_{-1}\Theta\right) + \frac{1}{n\sqrt{n}T^2} tr\left(Z^{0\prime}_{-1} Z^0_{-1}\Theta\right). \qquad (26)$$

Using similar arguments in the proof of Lemma 9(a), it is possible to show that

$$\frac{\left\|Z^0_{-1}\Theta\right\|}{\sqrt{n}T} = O_p(1).$$

So, by Lemma 9(a), we have

$$(26) \le \frac{1}{\sqrt{nT}} \frac{\|\alpha\|}{\sqrt{n}} \frac{\|l_T\|}{\sqrt{T}} \frac{\left\|Z^0_{-1}\Theta\right\|}{\sqrt{n}T} + \frac{1}{\sqrt{n}} \frac{\left\|Z^0_{-1}\right\|}{\sqrt{n}T} \frac{\left\|Z^0_{-1}\Theta\right\|}{\sqrt{n}T} = O_p\left(\frac{1}{\sqrt{n}}\right) = o_p(1).$$

Next, for part II_b, since

$$\begin{aligned}
II_b &= \frac{1}{nT} tr\left(\left(Z_{-1} - Z_{-1}(0)\right)' y\right) + \frac{1}{nT} tr\left(Z'_{-1}(0)\, y\right) \\
&= \frac{1}{nT} tr\left(Z'_{-1}(0)\, y\right) + o_p(1)
\end{aligned}$$

by Lemma 10(b), to have the required result, it is enough to show that

$$\frac{1}{nT} tr\left(Z'_{-1}(0)\, y\right) \Rightarrow \frac{1}{2} tr\left(B_f(1)\, B_f(1)'\, \Sigma_\beta\right) + \frac{1}{2}\omega_e^2 - \frac{1}{2} tr\left(\Sigma_f \Sigma_\beta\right) - \frac{1}{2}\sigma_e^2.$$

First, we have

$$\frac{1}{nT} tr\left(Z'_{-1}(0)\, y\right) = \frac{1}{nT} tr\left(\alpha l'_T y\right) + \frac{1}{nT} tr\left(Z^{0\prime}_{-1}(0)\, y\right) = \frac{1}{nT} tr\left(Z^{0\prime}_{-1}(0)\, y\right) + o_p(1),$$

where the last equality holds since

$$\frac{1}{nT} tr\left(\alpha l'_T y\right) \le \frac{1}{\sqrt{T}} \frac{\|\alpha\|}{\sqrt{n}} \frac{\|l'_T y\|}{\sqrt{nT}} = O_p\left(\frac{1}{\sqrt{T}}\right),$$

where the last equality holds by Lemma 9(i**).

Also, by definition, we may have

$$\frac{1}{nT}tr\left(Z_{-1}'(0)\,y\right)=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}z_{it-1}^{0}(0)\,y_{it}=\frac{1}{2nT}\sum_{i=1}^{n}\left(z_{iT}^{0}(0)^{2}-\sum_{t=1}^{T}y_{it}^{2}\right).$$

Using similar arguments to those in the proof of Part (a), we have

$$\begin{aligned}&\frac{1}{2nT}\sum_{i=1}^{n}z_{iT}^{0}(0)^{2}=\frac{1}{2nT}\sum_{i=1}^{n}\left(\beta_i^{0\prime}F_T+E_{iT}\right)^{2}\\ =\;&\frac{1}{2}tr\left(\frac{F_TF_T'}{T}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0}\beta_i^{0\prime}\right)+\frac{1}{2nT}\sum_{i=1}^{n}E_{iT}^{2}+O_p\left(\frac{1}{\sqrt{n}}\right)\\ \Rightarrow\;&\frac{1}{2}tr\left(B_f(1)\,B_f(1)'\,\Sigma_\beta\right)+\frac{1}{2}\omega_e^{2}.\end{aligned}$$

Next, under Assumptions 2, 3, 7, and 6, it is possible to show that

$$\begin{aligned}&\frac{1}{2nT}\sum_{i=1}^{n}\sum_{t=1}^{T}y_{it}^{2}\\ =\;&\frac{1}{2}tr\left(\frac{1}{T}\sum_{t=1}^{T}f_t^{0}f_t^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0}\beta_i^{0\prime}\right)+\frac{1}{2nT}\sum_{i=1}^{n}\sum_{t=1}^{T}e_{it}^{2}+\frac{1}{2\sqrt{n}}\frac{1}{T}\sum_{t=1}^{T}f_t^{0\prime}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^{0}e_{it}\right)\\ \to_p\;&\frac{1}{2}tr\left(\Sigma_f\Sigma_\beta\right)+\frac{1}{2}\sigma_e^{2}+o_p(1),\end{aligned}$$

where the $o_p(1)$ term holds because

$$\begin{aligned}\left|\frac{1}{2\sqrt{n}}\frac{1}{T}\sum_{t=1}^{T}f_t^{0\prime}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^{0}e_{it}\right)\right|&\le\frac{1}{2\sqrt{n}}\left(\frac{1}{T}\sum_{t=1}^{T}\left\|f_t^{0}\right\|^{2}\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\beta_i^{0}e_{it}\right\|^{2}\right)^{1/2}\\ &=\frac{1}{\sqrt{n}}O_p(1)\,O_p(1)\end{aligned}$$

by the Cauchy-Schwarz inequality, Assumption 7 and applying similar arguments used in the proof of Lemma 9(d), and the limit holds by Assumptions 7 and 6, and Lemma 6(c). Therefore, we have the required result,

$$\frac{1}{nT}tr\left(Z_{-1}'(0)\,y\right)\Rightarrow\frac{1}{2}tr\left(B_f(1)\,B_f(1)'\,\Sigma_\beta\right)+\frac{1}{2}\omega_e^{2}-\frac{1}{2}tr\left(\Sigma_f\Sigma_\beta\right)-\frac{1}{2}\sigma_e^{2},$$

and we complete the proof. ■

Proof of Lemma 2.

See Moon and Perron (2002). ■

Before we start the proof of Lemma 4, we introduce the following Lemmas. Define $\tilde{\Gamma}_i(j)=\frac{1}{T}\sum_t e_{it}e_{it+j}$, where the summation $\sum_t$ runs over $1\le t,t+j\le T$. Define $\tilde{\lambda}_{e,i}$ and $\tilde{\omega}_{e,i}^{2}$

$$\tilde{\lambda}_{e,i}=\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\tilde{\Gamma}_i(j)$$

and

$$\tilde{\omega}_{e,i}^2 = \sum_{j=-T+1}^{T-1} w\left(\frac{j}{h}\right)\tilde{\Gamma}_i(j).$$

We denote

$$\tilde{\lambda}_e^n = \frac{1}{n}\sum_{i=1}^n \tilde{\lambda}_{e,i},\ \tilde{\omega}_e^{n,2} = \frac{1}{n}\sum_{i=1}^n \tilde{\omega}_{e,i}^2,\ \text{and } \tilde{\phi}_e^{n,4} = \frac{1}{n}\sum_{i=1}^n \tilde{\omega}_{e,i}^4.$$

Lemma 16 *(a) Under the conditions of Lemma 4(a),* $\sqrt{n}\left(\hat{\lambda}_e^n - \tilde{\lambda}_e^n\right) = o_p(1)$.

(b) Under the conditions of Lemma 4(b), $\hat{\omega}_e^2 - \tilde{\omega}_e^2 = o_p(1)$.

(c) Under the conditions of Lemma 4(c), $\hat{\phi}_e^4 - \tilde{\phi}_e^4 = o_p(1)$.

Proof of Lemma 16.

Part (a).

By the definition of $\hat{e}$, we write

$$\begin{aligned}\hat{e} &= e - f^0\beta^{0\prime}\left(P_{\hat{\beta}_K} - P_{\beta^0}\right) - eP_{\hat{\beta}_K} - \left[\left(\hat{\rho}_{pool}-1\right)l_T\alpha' + \left(\hat{\rho}_{pool}-1\right)Z_{-1}^0 + Z_{-1}^0\left(\rho - I_n\right)\right]Q_{\hat{\beta}_K}\\ &= e - R_1 - R_2 - R_3,\ \text{say.}\end{aligned}$$

Define $R = R_1+R_2+R_3$. Let r_{it} be the $(t,i)^{th}$ element of the $(T\times n)$ matrix R. Similarly, define $r_{k,it}$ with R_k, $k=1,2,3$. Using this notation, we write

$$\begin{aligned}\sqrt{n}\left(\hat{\lambda}_e^n - \tilde{\lambda}_e^n\right) &= \frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sum_{i=1}^n\sum_{1\le t,t+j\le T}\left(\hat{e}_{it}\hat{e}_{it+j} - e_{it}e_{it+j}\right)\\ &= \frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sum_{i=1}^n\sum_{1\le t,t+j\le T}\left(r_{it}r_{it+j} - e_{it}r_{it+j} - r_{it}e_{it+j}\right)\\ &= I_a - II_a - III_a,\ \text{say.}\end{aligned}$$

For I_a, by the Cauchy Schwarz inequality,

$$I_a \le \sqrt{n}\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\right)\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T r_{it}^2$$

Under Assumption 12, since $\frac{1}{h_\lambda}\sum_{j=1}^T w\left(\frac{j}{h_\lambda}\right) \to \int_0^\infty w(x)\,dx < M$,

$$\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right) = O(h_\lambda). \tag{27}$$

Also,

$$\frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T r_{it}^2 = \frac{1}{nT}\left\|R_1+R_2+R_3\right\|^2 \le 2\left\{\frac{1}{nT}\left\|R_1+R_2\right\|^2 + \frac{1}{nT}\left\|R_3\right\|^2\right\}.$$

Since

$$\begin{aligned}R_1 + R_2 &= f^0\beta^{0\prime}\left(P_{\hat{\beta}_K} - P_{\beta^0}\right) + eP_{\hat{\beta}_K}\\ &= eP_{\beta^0} + y\left(P_{\hat{\beta}_K} - P_{\beta^0}\right),\end{aligned}$$

$$\left\|R_1+R_2\right\|^2 \leq 2\left\|eP_{\beta^0}\right\|^2+2\left\|y\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\right\|^2$$

First, since $\frac{\beta^{0\prime}e'e\beta^0}{nT}$ is positive definite,

$$\begin{aligned}\frac{1}{nT}\left\|eP_{\beta^0}\right\|^2 &= \frac{1}{n^2T}tr\left(e\beta^0\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}e'\right)=\frac{1}{n^2T}tr\left(\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}e'e\beta^0\right)\\ &\leq \frac{1}{n}\left\|\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\right\|tr\left(\frac{\beta^{0\prime}e'e\beta^0}{nT}\right).\end{aligned}$$

Since $\frac{\beta^{0\prime}\beta^0}{n}\rightarrow\Sigma_\beta>0$ by Assumption 6,

$$\left\|\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\right\|=O\left(1\right),$$

and by Lemma 9(g),

$$tr\left(\frac{\beta^{0\prime}e'e\beta^0}{nT}\right)=O_p\left(1\right).$$

Therefore,

$$\frac{1}{nT}\left\|eP_{\beta^0}\right\|^2=O_p\left(\frac{1}{n}\right). \tag{28}$$

Next, $\frac{1}{nT}\left\|y\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\right\|^2=O_p\left(\frac{1}{n}\right)$ because

$$\frac{1}{nT}\left\|y\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\right\|^2\leq\frac{1}{n}\frac{\|y\|^2}{nT}\left\|\sqrt{n}\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\right\|^2=O_p\left(\frac{1}{n}\right) \tag{29}$$

where the equality holds by Lemma 3 and the fact that $\frac{\|y\|^2}{nT}=O_p\left(1\right)$ by Lemma 9(k). Thus, from (28) and (29) we have

$$\frac{1}{nT}\left\|R_1+R_2\right\|^2=O_p\left(\frac{1}{n}\right). \tag{30}$$

For $\frac{1}{nT}\left\|R_3\right\|^2$, notice that

$$\begin{aligned}&\frac{1}{nT}\left\|R_3\right\|^2\\ \leq\ & 2T^2\left(\hat{\rho}_{pool}-1\right)^2\frac{1}{nT^3}\left\|l_T\alpha'Q_{\hat{\beta}_K}\right\|^2+2T^2\left(\hat{\rho}_{pool}-1\right)^2\frac{1}{nT^3}\left\|Z^0_{-1}Q_{\hat{\beta}_K}\right\|^2\\ &+2T^2\left\|\rho-I_n\right\|^2\frac{1}{nT^3}\left\|Z^0_{-1}Q_{\hat{\beta}_K}\right\|^2\\ \leq\ & \frac{2}{T}\left(T^2\left(\hat{\rho}_{pool}-1\right)^2\right)\left\{\frac{1}{T}\frac{\|l_T\|^2}{T}\frac{\|\alpha\|^2}{n}\left(1+\left\|P_{\hat{\beta}_K}\right\|\right)+\frac{\left\|Z^0_{-1}\right\|}{nT^2}\left(1+\left\|P_{\hat{\beta}_K}\right\|\right)\right\}\\ &+\frac{2}{T}\left(T^2\left\|\rho-I_n\right\|^2\right)\frac{\left\|Z^0_{-1}\right\|}{nT^2}\left(1+\left\|P_{\hat{\beta}_K}\right\|\right)\\ =\ & O_p\left(\frac{1}{T}\right),\end{aligned} \tag{31}$$

where the last line holds because $T\left(\hat{\rho}_{pool}-1\right)=O_p(1)$ by Lemma 1, $T\|\rho-I_n\|=O_p(1)$, $\frac{1}{nT^2}\left\|Z_{-1}^0\right\|^2=O_p(1)$ by Lemma 9(a), $\left\|P_{\hat{\beta}_K}\right\|=O_p(1)$ by Lemma 3, and $\left\|P_{\beta_K^*}\right\|^2=O_p(1)$.

Now, from (30) and (31), we deduce that

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}r_{it}^2=O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{T}\right). \tag{32}$$

Therefore,

$$I_a=\sqrt{n}O\left(h_\lambda\right)\left(O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{T}\right)\right)=O_p\left(\max\left(\frac{h_\lambda}{\sqrt{n}},\frac{h_\lambda\sqrt{n}}{T}\right)\right). \tag{33}$$

Next, for II_a, write

$$II_a=\frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\sum_{i=1}^{n}\sum_{1\le t,t+j\le T}e_{it}\left(r_{1,it+j}+r_{2,it+j}+r_{3,it+j}\right)=II_{aa}+II_{ab}+II_{ac},\text{ say.}$$

First, we have

$$\begin{aligned}
II_{aa} &= \frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\sum_{i=1}^{n}\sum_{1\le t,t+j\le T}e_{it}r_{1,it+j}=\frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\sum_{1\le t,t+j\le T}e_t'\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\beta^0 f_{t+j}^0\\
&= tr\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\left(\frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right)\\
&\le \sqrt{n}\left\|P_{\hat{\beta}_K}-P_{\beta^0}\right\|\left(\frac{1}{nT}\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\left\|\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right\|\right)\text{ since } tr(AB)\le\|A\|\,\|B\|.
\end{aligned}$$

From Lemma 3, we have

$$\sqrt{n}\left\|P_{\hat{\beta}_K}-P_{\beta^0}\right\|=O_p(1)$$

as $(n,T\to\infty)$ with $\frac{n}{T}\to 0$. Also,

$$\begin{aligned}
&\frac{1}{nT}\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\left\|\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right\|\\
&\le \left\|\frac{\beta^0}{\sqrt{n}}\right\|\left(\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\left\|\frac{1}{\sqrt{n}T}\sum_{1\le t,t+j\le T}f_{t+j}^0 e_t'\right\|\right)\\
&= \left\|\frac{\beta^0}{\sqrt{n}}\right\|\sum_{j=1}^{T-1}w\left(\frac{j}{h_\lambda}\right)\sqrt{\frac{1}{T^2}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T}tr\left(f_{t+j}^0 f_{s+j}^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n}e_{it}e_{is}\right)}.
\end{aligned}$$

Notice that

$$
\begin{aligned}
& E\left[\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sqrt{\frac{1}{T^2}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} tr\left(f_{t+j}^0 f_{s+j}^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^n e_{it}e_{is}\right)}\right] \\
\le & \sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sqrt{\frac{1}{T^2}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} trE\left(f_{t+j}^0 f_{s+j}^{0\prime}\right)E\left(\frac{1}{n}\sum_{i=1}^n e_{it}e_{is}\right)} \\
= & \frac{1}{\sqrt{T}}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sqrt{\frac{1}{T}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} tr\Gamma_f\left(t-s\right)\frac{1}{n}\sum_{i=1}^n \Gamma_{e,i}\left(t-s\right)} \\
\le & \frac{2}{\sqrt{T}}\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\right)\sqrt{\sum_{h=0}^{\infty}\left\|\Gamma_f\left(h\right)\right\|\bar{\Gamma}_e\left(h\right)} = O\left(\frac{h_\lambda}{\sqrt{T}}\right).
\end{aligned}
$$

Therefore,

$$
\frac{1}{nT}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\left\|\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right\| = O_p\left(\frac{h_\lambda}{\sqrt{T}}\right),
$$

and

$$
II_{aa} = O_p\left(\frac{h_\lambda}{\sqrt{T}}\right). \tag{34}
$$

Next, similarly to II_{aa} and letting $\hat{\beta}_{K,i}'$ be the i^{th} row of $\hat{\beta}_K$, we have

$$
\begin{aligned}
II_{ab} = & \frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sum_{1\le t,t+j\le T} e_t' r_{2,t+j} = \frac{1}{\sqrt{n}T}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\sum_{1\le t,t+j\le T} e_t' P_{\hat{\beta}_K} e_{t+j} \\
= & \frac{1}{\sqrt{n}} tr\left(\frac{1}{n}\sum_{i=1}^n \hat{\beta}_{Ki}\hat{\beta}_{Ki}'\right)^{-1}\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\frac{1}{T}\sum_{1\le t,t+j\le T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n e_{it+j}\hat{\beta}_{K,i}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n e_{it}\hat{\beta}_{K,i}'\right)\right) \\
\le & \frac{1}{\sqrt{n}}\left\|\left(\frac{\hat{\beta}_K'\hat{\beta}_K}{n}\right)^{-1}\right\|\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\frac{1}{T}\sum_{1\le t,t+j\le T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n e_{it+j}\hat{\beta}_{K,i}\right\|\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n e_{it}\hat{\beta}_{K,i}\right\|\right) \\
\le & \frac{1}{\sqrt{n}}\left\|\left(\frac{\hat{\beta}_K'\hat{\beta}_K}{n}\right)^{-1}\right\|\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\right)\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^n e_{it}\hat{\beta}_{K,i}\right\|^2 \\
= & \frac{1}{\sqrt{n}}\left\|\left(\frac{\hat{\beta}_K'\hat{\beta}_K}{n}\right)^{-1}\right\|\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\right) tr\left(\frac{1}{nT}\hat{\beta}_K' e' e\hat{\beta}_K\right),
\end{aligned} \tag{35}
$$

where the first inequality holds because $tr(AB) \le \|A\|\,\|B\|$ and the second inequality holds by the Cauchy-Schwarz inequality. Since

$$
\frac{\hat{\beta}_K'\hat{\beta}_K}{n} = \frac{\bar{\beta}_K}{\sqrt{n}}'\left(\frac{\hat{y}'\hat{y}}{nT}\right)\frac{\bar{\beta}_K}{\sqrt{n}} = \Lambda_{nT,K} \to_p \Lambda_K,
$$

where Λ_K is of full rank by Lemma 13(a),

$$\left\| \left(\frac{\hat{\beta}_K' \hat{\beta}_K}{n} \right)^{-1} \right\| = O_p(1).$$

By Assumption 11, we have

$$\sum_{j=1}^{T} w\left(\frac{j}{h_\lambda}\right) = O(h_\lambda).$$

Finally,

$$\begin{aligned} tr\left(\frac{1}{nT}\hat{\beta}_K' e' e \hat{\beta}_K\right) &= \frac{1}{nT}\left\|e\hat{\beta}_K\right\|^2 \leq 2\frac{1}{nT}\|e\|^2 \left\|\hat{\beta}_K - \beta_K^*\right\|^2 + 2\frac{1}{nT}\left\|e\beta^0\right\|^2 \|H_K\|^2 \\ &= O_p(1) O_p(1) + O_p(1) O_p(1) = O_p(1), \end{aligned}$$

where the first inequality of the second line holds since $\sup_{it} Ee_{it}^2 < M$, by Lemma 15(b), Lemma 9(j), and Lemma 14. Therefore, $(35) = O_p\left(\frac{h_\lambda}{\sqrt{n}}\right)$, and we have

$$II_{ab} = O_p\left(\frac{h_\lambda}{\sqrt{n}}\right). \tag{36}$$

Next, for II_{ac}, using the Cauchy-Schwarz inequality, we have

$$II_{ac} \leq \sqrt{n}\left(\sum_{j=1}^{T-1} w\left(\frac{j}{h_\lambda}\right)\right)\sqrt{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} e_{it}^2}\sqrt{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} r_{3,it}^2}.$$

Since $\sup_{it} Ee_{it}^2 < M$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} e_{it}^2 = O_p(1)$. Also, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} r_{3,it} = \frac{1}{nT}\|R_3\|^2 = O_p\left(\frac{1}{T}\right)$ by (31). Thus,

$$II_{ac} = \sqrt{n} O(h_\lambda) O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(\sqrt{\frac{nh_\lambda^2}{T}}\right). \tag{37}$$

In view of (34), (36), and (37) and since $\frac{n}{T} \to 0$ under Assumption 10, we have

$$II_a = O_p\left(\frac{h_\lambda}{\sqrt{T}}\right) + O_p\left(\frac{h_\lambda}{\sqrt{n}}\right) + O_p\left(\sqrt{\frac{nh_\lambda^2}{T}}\right) = O_p\left(\max\left\{\frac{h_\lambda}{\sqrt{n}}, \sqrt{\frac{nh_\lambda^2}{T}}\right\}\right).$$

In a similar fashion, we can show that

$$III_a = O_p\left(\max\left\{\frac{h_\lambda}{\sqrt{n}}, \sqrt{\frac{nh_\lambda^2}{T}}\right\}\right).$$

Combining I_a, II_a, and III_a, we have

$$\begin{aligned} \sqrt{n}\left(\hat{\lambda}_e^n - \tilde{\lambda}_e^n\right) &= O_p\left(\max\left(\frac{h_\lambda}{\sqrt{n}}, \frac{h_\lambda\sqrt{n}}{T}\right)\right) + O_p\left(\max\left\{\frac{h_\lambda}{\sqrt{n}}, \sqrt{\frac{nh_\lambda^2}{T}}\right\}\right) \\ &= O_p\left(\max\left(\frac{h_\lambda}{\sqrt{n}}, \frac{h_\lambda\sqrt{n}}{\sqrt{T}}\right)\right). \end{aligned}$$

Under the regularity conditions of the lemma,

$$\frac{h_\lambda}{\sqrt{n}} \sim n^{b-1/2} \to 0$$

since $b < \frac{1}{2}$. Next, for n, T large,

$$\begin{aligned} \frac{h_\lambda \sqrt{n}}{\sqrt{T}} &\sim \frac{n^{1/2+b}}{\sqrt{T}} = e^{\log \frac{n^{1/2+b}}{\sqrt{T}}} = e^{1/2\left(1+2b-\frac{\log T}{\log n}\right)\log n} = n^{1/2\left(1+2b-\frac{\log T}{\log n}\right)} \\ &\leq n^{1/2\left(1+2b-\liminf \frac{\log T}{\log n}\right)} \to 0 \end{aligned}$$

since $b < \frac{a-1}{2}$. Therefore,

$$\sqrt{n}\left(\hat{\lambda}_e^n - \tilde{\lambda}_e^n\right) = O_p\left(\max\left(\frac{h_p}{\sqrt{n}}, \frac{h_p\sqrt{n}}{\sqrt{T}}\right)\right) = o_p(1),$$

as required. ■

Part (b)

Similarly to Part (a), we write

$$\begin{aligned} \hat{e} &= e - f^0\beta^{0\prime}\left(P_{\hat{\beta}_K} - P_{\beta^0}\right) - eP_{\hat{\beta}_K} - \left[\left(\hat{\rho}_{pool}-1\right)l_T\alpha' + \left(\hat{\rho}_{pool}-1\right)Z_{-1}^0 + Z_{-1}^0\left(\rho - I_n\right)\right]Q_{\hat{\beta}_K} \\ &= e - R_1 - R_2 - R_3, \text{ say.} \end{aligned} \quad (38)$$

Using this notation, we write

$$\begin{aligned} \hat{\omega}_e^2 - \tilde{\omega}_e^2 &= \frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{i=1}^{n}\sum_{1\leq t,t+j\leq T}\left(\hat{e}_{it}\hat{e}_{it+j} - e_{it}e_{it+j}\right) \\ &= \frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{i=1}^{n}\sum_{1\leq t,t+j\leq T}\left(r_{it}r_{it+j} - e_{it}r_{it+j} - r_{it}e_{it+j}\right) \\ &= I_b - II_b - III_b, \text{ say.} \end{aligned}$$

For I_b, by the Cauchy Schwarz inequality,

$$I_b \leq \left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\right)\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} r_{it}^2$$

Under Assumption 11, since $\frac{1}{h_\omega}\sum_{j=-T}^{T} w\left(\frac{j}{h_\omega}\right) \to \int_{-\infty}^{\infty} w(x)\,dx < M$,

$$\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right) = O(h_\omega).$$

Also, as shown in Part (a),

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} r_{it}^2 = O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{T}\right).$$

Therefore,

$$I_b = O(h_\omega)\left(O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{T}\right)\right) = O_p\left(\max\left(\frac{h_\omega}{n}, \frac{h_\omega}{T}\right)\right). \quad (39)$$

Next, for II_b, write

$$II_b = \frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{i=1}^{n}\sum_{1\le t,t+j\le T} e_{it}\left(r_{1,it+j}+r_{2,it+j}+r_{3,it+j}\right) = II_{ba}+II_{bb}+II_{bc}, \text{ say.}$$

First, we have

$$\begin{aligned} II_{ba} &= \frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{i=1}^{n}\sum_{1\le t,t+j\le T} e_{it}r_{1,it+j} = \frac{1}{nT}\sum_{j=1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{1\le t,t+j\le T} e_t'\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\beta^0 f_{t+j}^0 \\ &= tr\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)\left(\frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right) \\ &\le \frac{1}{\sqrt{n}}\sqrt{n}\left\|P_{\hat{\beta}_K}-P_{\beta^0}\right\|\left(\frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\left\|\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right\|\right) \text{ since } tr\left(AB\right)\le\|A\|\|B\|. \end{aligned}$$

By Lemma 3, we have

$$\sqrt{n}\left\|P_{\hat{\beta}_K}-P_{\beta^0}\right\| = O_p\left(1\right)$$

as $(n,T\to\infty)$ with $\frac{n}{T}\to 0$. Also,

$$\begin{aligned} &\frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\left\|\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right\| \\ \le\ & \left\|\frac{\beta^0}{\sqrt{n}}\right\|\left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\left\|\frac{1}{\sqrt{n}T}\sum_{1\le t,t+j\le T} f_{t+j}^0 e_t'\right\|\right) \\ =\ & \left\|\frac{\beta^0}{\sqrt{n}}\right\|\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sqrt{\frac{1}{T^2}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} tr\left(f_{t+j}^0 f_{s+j}^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n} e_{it}e_{is}\right)}. \end{aligned}$$

Similar to Part (a), for some constant M,

$$\begin{aligned} &E\left[\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sqrt{\frac{1}{T^2}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} tr\left(f_{t+j}^0 f_{s+j}^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n} e_{it}e_{is}\right)}\right] \\ \le\ & \sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sqrt{\frac{1}{T^2}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} trE\left(f_{t+j}^0 f_{s+j}^{0\prime}\right)E\left(\frac{1}{n}\sum_{i=1}^{n} e_{it}e_{is}\right)} \\ =\ & \frac{1}{\sqrt{T}}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sqrt{\frac{1}{T}\sum_{1\le t,t+j\le T}\sum_{1\le s,s+j\le T} tr\Gamma_f\left(t-s\right)\left(\frac{1}{n}\sum_{i=1}^{n}\Gamma_{e,i}\left(t-s\right)\right)} \\ \le\ & \frac{M}{\sqrt{T}}\left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\right)\sqrt{\sum_{h=0}^{\infty}\left\|\Gamma_f\left(h\right)\right\|\bar{\Gamma}_e\left(h\right)} = O\left(\frac{h_\omega}{\sqrt{T}}\right), \end{aligned}$$

which leads to

$$\frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\left\|\sum_{1\le t,t+j\le T}\beta^0 f_{t+j}^0 e_t'\right\| = O_p\left(\frac{h_\omega}{\sqrt{T}}\right).$$

Hence,

$$II_{ba} = O_p\left(\frac{h_\omega}{\sqrt{nT}}\right). \tag{40}$$

Next, similar to II_{ba} and Part (a), we have

$$\begin{aligned}
II_{bb} &= \frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{1\le t,t+j\le T} e_t' r_{2,t+j} = \frac{1}{nT}\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\sum_{1\le t,t+j\le T} e_t' P_{\hat{\beta}_K} e_{t+j} \\
&= \frac{1}{n} tr\left[\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\beta}_{K,i}\hat{\beta}_{K,i}'\right)^{-1}\left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\frac{1}{T}\sum_{1\le t,t+j\le T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{it+j}\hat{\beta}_{K,i}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{it}\hat{\beta}_{K,i}'\right)\right)\right] \\
&\le \frac{1}{n}\left\|\left(\frac{\hat{\beta}_K'\hat{\beta}_K}{n}\right)^{-1}\right\|\left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\frac{1}{T}\sum_{1\le t,t+j\le T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{it+j}\hat{\beta}_{K,i}\right\|\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{it}\hat{\beta}_{K,i}\right\|\right) \\
&\le \frac{1}{n}\left\|\left(\frac{\hat{\beta}_K'\hat{\beta}_K}{n}\right)^{-1}\right\|\left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\right)\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{it}\hat{\beta}_{K,i}\right\|^2 \\
&= \frac{1}{n}\left\|\left(\frac{\hat{\beta}_K'\hat{\beta}_K}{n}\right)^{-1}\right\|\left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\right) tr\left(\frac{1}{nT}\hat{\beta}_K' e' e\hat{\beta}_K\right) \\
&= \frac{1}{n} O_p(1)\, O(h_\omega)\, O_p(1).
\end{aligned}$$

So, we have

$$II_{bb} = O_p\left(\frac{h_\omega}{n}\right). \tag{41}$$

Next, for II_{bc}, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
II_{bc} &\le \left(\sum_{j=-T+1}^{T-1} w\left(\frac{j}{h_\omega}\right)\right)\sqrt{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} e_{it}^2}\sqrt{\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} r_{3,it}^2} \\
&= O(h_\omega)\, O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(\sqrt{\frac{h_\omega^2}{T}}\right)
\end{aligned} \tag{42}$$

In view of (40), (41), and (42) and since $\frac{n}{T} \to 0$ under Assumption 10, we have

$$II_b = O_p\left(\frac{h_\omega}{\sqrt{nT}}\right) + O_p\left(\frac{h_\omega}{n}\right) + O_p\left(\sqrt{\frac{h_\omega^2}{T}}\right) = O_p\left(\max\left\{\frac{h_\omega}{n}, \sqrt{\frac{h_\omega^2}{T}}\right\}\right). \tag{43}$$

in a similar fashion, we can show that

$$III_b = O_p\left(\max\left\{\frac{h_\omega}{n}, \sqrt{\frac{h_\omega^2}{T}}\right\}\right).$$

Combining I_b, II_b, and III_b, we have

$$\hat{\omega}_e^2 - \tilde{\omega}_e^2 = O_p\left(\max\left(\frac{h_\omega}{n}, \frac{h_\omega}{T}\right)\right) + O_p\left(\max\left\{\frac{h_\omega}{n}, \sqrt{\frac{h_\omega^2}{T}}\right\}\right) = O_p\left(\max\left(\frac{h_\omega}{n}, \frac{h_\omega}{\sqrt{T}}\right)\right).$$

Under the regularity conditions of Part (b), since $b < 1$,

$$\frac{h_\omega}{n} \sim n^{b-1} \to 0.$$

Also, by definition, for n, T large,

$$\frac{h_\omega}{\sqrt{T}} \sim \frac{n^b}{\sqrt{T}} = e^{\log \frac{n^b}{\sqrt{T}}} = e^{\frac{1}{2}\left(2b - \frac{\log T}{\log n}\right)\log n} = n^{\frac{1}{2}\left(2b - \frac{\log T}{\log n}\right)} \leq n^{\frac{1}{2}\left(2b - \liminf \frac{\log T}{\log n}\right)} \to 0 \quad (44)$$

because $2b < a = \liminf \frac{\log T}{\log n}$.

Therefore, $O_p\left(\max\left(\frac{h_\omega}{n}, \frac{h_\omega}{\sqrt{T}}\right)\right) = o_p(1)$, and we have

$$\hat{\omega}_e^2 - \tilde{\omega}_e^2 = o_p(1),$$

as required. ■

Part (c)

Notice by the Cauchy-Schwarz inequality that

$$\begin{aligned} \left|\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^4 - \tilde{\omega}_{e,i}^4\right)\right| &= \left|\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 - \tilde{\omega}_{e,i}^2\right)\left(\hat{\omega}_{e,i}^2 + \tilde{\omega}_{e,i}^2\right)\right| \\ \leq & \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 - \tilde{\omega}_{e,i}^2\right)^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 + \tilde{\omega}_{e,i}^2\right)^2} \\ \leq & \frac{\sqrt{2}}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 - \tilde{\omega}_{e,i}^2\right)^2 + \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 - \tilde{\omega}_{e,i}^2\right)^2}\sqrt{\frac{8}{n}\sum_{i=1}^{n}\tilde{\omega}_{e,i}^4}, \end{aligned} \quad (45)$$

where the last inequality holds because $(a+2b)^2 \leq 2a^2 + 8b^2$ and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. In view of (45), the required result follows if we show that

$$(\mathrm{c}_1)\ \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 - \tilde{\omega}_{e,i}^2\right)^2 = o_p(1) \quad (46)$$

and

$$(\mathrm{c}_2)\ \frac{1}{n}\sum_{i=1}^{n}\tilde{\omega}_{e,i}^4 = O_p(1). \quad (47)$$

Part ($\mathbf{c}_2$)

Notice that

$$\frac{1}{n}\sum_{i=1}^{n}\tilde{\omega}_{e,i}^4 = \frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^4 - \omega_{e,i}^4\right) + \frac{1}{n}\sum_{i=1}^{n}\omega_{e,i}^4, \quad (48)$$

Since $\sup_i \omega_{e,i}^4 \leq \left(\sum_{j=0}^{\infty}\bar{d}_j\right)^4 < M$, we have

$$\frac{1}{n}\sum_{i=1}^{n}\omega_{e,i}^4 = O(1). \quad (49)$$

Similar to (45), we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{4}-\omega_{e,i}^{4}\right) &\leq \frac{\sqrt{2}}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{2}-\omega_{e,i}^{2}\right)^{2}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{2}-\omega_{e,i}^{2}\right)^{2}}\sqrt{\frac{8}{n}\sum_{i=1}^{n}\omega_{e,i}^{4}} \\ &= \frac{\sqrt{2}}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{2}-\omega_{e,i}^{2}\right)^{2}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{2}-\omega_{e,i}^{2}\right)^{2}}O(1). \qquad (50)\end{aligned}$$

Since

$$\sup_i E\left(\tilde{\omega}_{e,i}^{2}-\omega_{e,i}^{2}\right)^{2}\leq \sup_i Var\left(\tilde{\omega}_{e,i}^{2}\right)+\sup_i\left(bias\left(\tilde{\omega}_{e,i}^{2}\right)\right)^{2}=O\left(\frac{h_\phi}{T}\right)+O\left(\frac{1}{h_\phi^{2q}}\right),$$

if $\frac{h_\phi^q}{T}\rightarrow 0$ when $q\geq 1$.

Using similar arguments used in (44), when $0<q<1$, for n,T large,

$$\frac{h_\phi}{T}\sim\frac{n^b}{T}=e^{\left(b-\frac{\log T}{\log n}\right)\log n}=n^{\left(b-\frac{\log T}{\log n}\right)}\leq n^{\left(b-\liminf\frac{\log T}{\log n}\right)}\rightarrow 0$$

since $b<\frac{1}{4}<a$, where $a=\liminf\frac{\log T}{\log n}>1$. Similarly, if $q\geq 1$, for n,T large,

$$\frac{h_\phi^q}{T}\sim\frac{n^{bq}}{T}=e^{\left(bq-\frac{\log T}{\log n}\right)\log n}\leq n^{(bq-\liminf\frac{\log T}{\log n})}\rightarrow 0$$

since $b<\min\left\{\frac{1}{4},\frac{a}{q}\right\}$. Also, $\frac{1}{h_\phi^{2q}}\rightarrow 0$ since $q>0$. Therefore, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{2}-\omega_{e,i}^{2}\right)^{2}=o_p(1),$$

and from (50), we have

$$\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^{4}-\omega_{e,i}^{4}\right)=o_p(1). \qquad (51)$$

Thus, in view of (48) together with (49) and (51), we have the required result for Part (c_2),

$$\frac{1}{n}\sum_{i=1}^{n}\tilde{\omega}_{e,i}^{4}=O_p(1).\ \blacksquare$$

Part (c_1)

Next, for part (c_1), from the decomposition

$$\begin{aligned}\hat{e} &= e-f^0\beta^{0\prime}\left(P_{\hat{\beta}_K}-P_{\beta^0}\right)-eP_{\hat{\beta}_K}-\left[\left(\hat{\rho}_{pool}-1\right)l_T\alpha'+\left(\hat{\rho}_{pool}-1\right)Z_{-1}^0+Z_{-1}^0\left(\rho-I_n\right)\right]Q_{\hat{\beta}_K} \\ &= e-R,\end{aligned}$$

and letting r_{it} be the $(t,i)^{th}$ element of the $(T\times n)$ matrix R, we have by the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^{2}-\tilde{\omega}_{e,i}^{2}\right)^{2}\\
=\ &\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\frac{1}{T}\sum_{1\leq t,t+j\leq T}\left(\hat{e}_{it}\hat{e}_{it+j}-e_{it}e_{it+j}\right)\right]^{2}\\
=\ &\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\frac{1}{T}\sum_{1\leq t,t+j\leq T}\left(r_{it}r_{it+j}-e_{it}r_{it+j}-r_{it}e_{it+j}\right)\right]^{2}\\
\leq\ &\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\left\{\left|\frac{1}{T}\sum_{1\leq t,t+j\leq T}r_{it}r_{it+j}\right|+\left|\frac{1}{T}\sum_{1\leq t,t+j\leq T}e_{it}r_{it+j}\right|+\left|\frac{1}{T}\sum_{1\leq t,t+j\leq T}r_{it}e_{it+j}\right|\right\}\right]^{2}\\
\leq\ &\frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\left\{\begin{array}{c}\sqrt{\frac{1}{T}\sum_{1\leq t\leq T}r_{it}^{2}}\sqrt{\frac{1}{T}\sum_{1\leq t+j\leq T}r_{it+j}^{2}}+\\ \sqrt{\frac{1}{T}\sum_{1\leq t\leq T}e_{it}^{2}}\sqrt{\frac{1}{T}\sum_{1\leq t+j\leq T}r_{it+j}^{2}}\\ +\sqrt{\frac{1}{T}\sum_{1\leq t\leq T}r_{it}^{2}}\sqrt{\frac{1}{T}\sum_{1\leq t+j\leq T}e_{it+j}^{2}}\end{array}\right\}\right]^{2}\\
\leq\ &\left(\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\right)^{2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}r_{it}^{2}\right)^{2}+2\left(\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\right)^{2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}r_{it}^{2}\right)\left(\frac{1}{T}\sum_{t=1}^{T}e_{it}^{2}\right)\\
\leq\ &2\left(\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)\right)^{2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}r_{it}^{2}\right)^{2}}\left\{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}r_{it}^{2}\right)^{2}}+\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}e_{it}^{2}\right)^{2}}\right\}.
\end{aligned}\tag{52}$$

Notice that

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}r_{it}^{2}\right)^{2}}\leq\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}r_{it}^{2}\right)=O_{p}\left(\max\left\{\frac{1}{\sqrt{n}},\frac{\sqrt{n}}{T}\right\}\right),$$

where the last equality holds by (32). Also, by the Cauchy-Schwarz inequality,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}e_{it}^{2}\right)^{2}\leq\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}e_{it}^{4}=O_{p}\left(1\right)$$

since $\sup_{it}Ee_{it}^{4}<M$ by Lemma 6(a). Because $\sum_{j=-T+1}^{T-1}w\left(\frac{j}{h_{\phi}}\right)=O\left(h_{\phi}\right),$ in view of (52) we have

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^{2}-\tilde{\omega}_{e,i}^{2}\right)^{2}=O\left(h_{\phi}^{2}\right)O_{p}\left(\max\left\{\frac{1}{\sqrt{n}},\frac{\sqrt{n}}{T}\right\}\right)=O_{p}\left(\max\left\{\frac{h_{\phi}^{2}}{\sqrt{n}},\frac{h_{\phi}^{2}\sqrt{n}}{T}\right\}\right).$$

Since $b<\frac{1}{4}$,

$$\frac{h_{\phi}^{2}}{\sqrt{n}}\sim n^{2b-1/2}\rightarrow 0.$$

Also,

$$\frac{h_\phi^2\sqrt{n}}{T} \sim \frac{n^{2b+1/2}}{T} = e^{\left(2b+\frac{1}{2}-\frac{\log T}{\log n}\right)\log n} \leq n^{\left(2b+\frac{1}{2}-\liminf\frac{\log T}{\log n}\right)} \to 0$$

since $b < \frac{1}{4} < \frac{a}{2} - \frac{1}{4}$ (recall $a = \liminf \frac{\log T}{\log n} > 1$). Therefore,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{\omega}_{e,i}^2 - \tilde{\omega}_{e,i}^2\right)^2 = o_p(1),$$

and this completes the proof of Part (c_1). ■

Proof of Lemma 4.
Part (a).

In order to have the required result, by Lemma 16(a), it is enough to show that

$$\sqrt{n}\left(\tilde{\lambda}_e^n - \lambda_e^n\right) = o_p(1).$$

Along the same lines as the proofs of Theorems 9 and 10 of Hannan (1970), it is possible to show under Assumption 2 that

$$E\left(\sqrt{n}\left(\tilde{\lambda}_e^n - \lambda_e^n\right)\right)^2 \leq \sup_i var\left(\tilde{\lambda}_{e,i}\right) + n\left(\sup_i bias\left(\tilde{\lambda}_{e,i}\right)\right)^2 = O\left(\frac{h_\lambda}{T}\right) + O\left(\frac{n}{h_\lambda^{2q}}\right),$$

if $\frac{h_\lambda^q}{T} \to 0$ in the case of $q \geq 1$[3]. So, we have

$$\sqrt{n}\left(\tilde{\lambda}_e^n - \lambda_e^n\right) = O_p\left(\max\left\{\sqrt{\frac{h_\lambda}{T}}, \sqrt{\frac{n}{h_\lambda^{2q}}}\right\}\right),$$

if $\frac{h_\lambda^q}{T} \to 0$ with $q \geq 1$

First, since $\frac{1}{2q} < b$,

$$\frac{n}{h_\lambda^{2q}} \sim n^{(1-2qb)} \to 0.$$

Next, for n, T large,

$$\frac{h_\lambda^q}{T} \sim \frac{n^{bq}}{T} = e^{\log\left(\frac{n^{bq}}{T}\right)} = e^{\left(bq-\frac{\log T}{\log n}\right)\log n} = n^{bq-\frac{\log T}{\log n}} \leq n^{bq-\liminf\frac{\log T}{\log n}} \to 0$$

since $b < \frac{a}{q}$ if $q \geq 1$. Similarly, $\frac{h_\lambda}{T} \to 0$. Therefore,

$$\sqrt{n}\left(\tilde{\lambda}_e^n - \lambda_e^n\right) = O_p\left(\max\left\{\sqrt{\frac{h_\lambda}{T}}, \sqrt{\frac{n}{h_\lambda^{2q}}}\right\}\right) = o_p(1),$$

as required for Part (a). ■

Part (b).

[3] For details on this, see equation (B.27) on page 985 of Moon and Phillips (2000).

Similar to Part (a), by Lemma 16(b), it is sufficient to show that

$$\tilde{\omega}_e^{2,n} - \omega_e^{2,n} = o_p(1).$$

Along the same lines as the proofs of Theorems 9 and 10 of Hannan (1970), it is possible to show under Assumption 2 that

$$\begin{aligned} E\left(\tilde{\omega}_e^{2,n} - \omega_e^{2,n}\right)^2 &= E\left(\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\omega}_{e,i}^2 - E\tilde{\omega}_{e,i}^2\right)\right)^2 + \left(\frac{1}{n}\sum_{i=1}^{n}\left(E\tilde{\omega}_{e,i}^2 - \omega_{e,i}^2\right)\right)^2 \\ &\leq \frac{1}{n}\sup_i var\left(\tilde{\omega}_{e,i}^2\right) + \left(\sup_i bias\left(\tilde{\omega}_{e,i}^2\right)\right)^2 = O\left(\frac{h_\omega}{nT}\right) + O\left(\frac{1}{h_\omega^{2q}}\right), \end{aligned}$$

if $\frac{h_\omega^q}{T} \to 0$ in the case of $q \geq 1$. So, we have

$$\tilde{\omega}_e^{2,n} - \omega_e^{2,n} = O_p\left(\max\left\{\sqrt{\frac{h_\omega}{nT}}, \frac{1}{h_\omega^q}\right\}\right).$$

Under the conditions in the lemma,

$$\frac{1}{h_\omega^q} \to 0$$

and for n, T large,

$$\frac{h_\omega}{nT} \sim \frac{n^{b-1}}{T} = e^{\log\left(\frac{n^{b-1}}{T}\right)} = e^{\left(b-1-\frac{\log T}{\log n}\right)\log n} = n^{b-1-\frac{\log T}{\log n}} \leq n^{b-1-\liminf \frac{\log T}{\log n}} \to 0$$

since $b < 1 < a+1$. Also, if $q \geq 1$

$$\frac{h_\omega^q}{T} \sim \frac{n^{bq}}{T} = e^{\log\left(\frac{n^{bq}}{T}\right)} = e^{\left(bq-\frac{\log T}{\log n}\right)\log n} = n^{bq-\frac{\log T}{\log n}} \leq n^{bq-\liminf \frac{\log T}{\log n}}$$

since $q < \min\left\{1, \frac{a}{2}, \frac{a}{q}\right\}$, $a = \liminf \frac{\log T}{\log n}$. Therefore,

$$\tilde{\omega}_e^{2,n} - \omega_e^{2,n} = o_p(1),$$

and we have all the required results to complete the proof. ■

Proof of Theorem 2 : Parts (i) – (iv) in the proof of Part (b) in Moon and Perron (2003).

We need to show that

$$\text{(i)} \quad \left\|\frac{\hat{\beta}_K' y' l_T \alpha' \hat{\beta}_K - \beta_K^{*\prime} y' l_T \alpha' \beta_K^*}{n\sqrt{n}T}\right\| = o_p(1),$$

$$\text{(ii)} \quad \left\|\frac{\hat{\beta}_K'(\rho - I_n) Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n)\hat{\beta}_K - \beta_K^{*\prime}(\rho - I_n) Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n)\beta_K^*}{n\sqrt{n}T}\right\| = o_p(1),$$

$$\text{(iii)} \quad \left\|\frac{\hat{\beta}_K' Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n)\hat{\beta}_K - \beta_K^{*\prime} Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n)\beta_K^*}{n\sqrt{n}T}\right\| = o_p(1)$$

$$\text{(iv)} \quad \left\|\frac{\hat{\beta}_K'(\rho - I_n) Z_{-1}^{0\prime} y \hat{\beta}_K - \beta_K^{*\prime}(\rho - I_n) Z_{-1}^{0\prime} y \beta_K^*}{n\sqrt{n}T}\right\| = o_p(1),$$

$$\text{(v)} \quad \left\|\frac{\hat{\beta}_K' Z_T^0 Z_T^{0\prime}\hat{\beta}_K - \beta_K^{*\prime} Z_T^0 Z_T^{0\prime}\beta_K^*}{n\sqrt{n}T}\right\| = o_p(1),$$

$$\text{and (vi)} \quad \left\|\frac{\hat{\beta}_K' y' y \hat{\beta}_K - \beta_K^{*\prime} y' y \beta_K^*}{n\sqrt{n}T}\right\| = o_p(1).$$

Part (i) holds because

$$\begin{aligned}
&\left\| \frac{\hat{\beta}_K' y' l_T \alpha' \hat{\beta}_K - \beta_K^{*\prime} y' l_T \alpha' \beta_K^*}{n\sqrt{n}T} \right\| \\
\leq\ & \left\| \frac{\left(\hat{\beta}_K - \beta_K^*\right)' y' l_T \alpha' \hat{\beta}_K}{n\sqrt{n}T} \right\| + \left\| \frac{\beta_K^{*\prime} y' l_T \alpha' \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T} \right\| \\
\leq\ & \frac{2}{\sqrt{T}} \left\| \hat{\beta}_K - \beta_K^* \right\| \frac{\|y' l_T\|}{\sqrt{nT}} \frac{\|\alpha\|}{\sqrt{n}} \left(\frac{\left\|\hat{\beta}_K\right\|}{\sqrt{n}} + \frac{\|\beta_K^*\|}{\sqrt{n}} \right) = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1),
\end{aligned}$$

where the first equality holds by Lemma 15(b) and Lemma 9(i**).

Part (ii) holds because

$$\begin{aligned}
&\left\| \frac{\hat{\beta}_K' (\rho - I_n) Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \hat{\beta}_K - \beta_K^{*\prime} (\rho - I_n) Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \beta_K^*}{n\sqrt{n}T} \right\| \\
\leq\ & \frac{\sqrt{n}}{T} \|T(\rho - I_n)\|^2 \frac{\left\|Z_{-1}^0\right\|^2}{nT^2} \left\{ \frac{\left\|\hat{\beta}_K\right\|^2 + \|\beta_K^*\|^2}{n} \right\} = O_p\left(\frac{\sqrt{n}}{T}\right) = o_p(1).
\end{aligned}$$

For Part (iii), by the triangle inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\| \frac{\hat{\beta}_K' Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \hat{\beta}_K - \beta_K^{*\prime} Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \beta_K^*}{n\sqrt{n}T} \right\| \\
\leq\ & \left\| \frac{\left(\hat{\beta}_K - \beta_K^*\right)' Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T} \right\| + \left\| \frac{\left(\hat{\beta}_K - \beta_K^*\right)' Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \beta_K^*}{n\sqrt{n}T} \right\| \\
&+ \left\| \frac{\beta_K^{*\prime} Z_{-1}^{0\prime} Z_{-1}^0 (\rho - I_n) \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T} \right\| \\
\leq\ & \frac{\left\|Z_{-1}^0\right\|^2}{nT^2} \left\{ \begin{array}{c} \frac{1}{n} \left\| \hat{\beta}_K - \beta_K^* \right\| \left\| \Theta \left(\hat{\beta}_K - \beta_K^*\right) \right\| \\ + \frac{1}{\sqrt{n}} \left(\left\| \hat{\beta}_K - \beta_K^* \right\| \frac{\|\Theta \beta_K^*\|}{\sqrt{n}} + \frac{\|\beta_K^*\|}{\sqrt{n}} \left\| \Theta \left(\hat{\beta}_K - \beta_K^*\right) \right\| \right) \end{array} \right\}. \qquad (53)
\end{aligned}$$

By modifying the proof of Lemma 15(a) and (b), it is possible to show that

$$\left\| \Theta \left(\hat{\beta}_K - \beta_K^*\right) \right\| = O_p(1).$$

Also,

$$\frac{\|\Theta \beta_K^*\|}{\sqrt{n}} \leq \frac{\left\|\Theta \beta^0\right\|}{\sqrt{n}} \|H_K\| = O_p(1).$$

Thus, together with Lemma 9(a), we have

$$(53) = O_p(1) \left\{ \frac{1}{n} O_p(1) + \frac{1}{\sqrt{n}} O_p(1) \right\} = o_p(1),$$

as required for Part (iii).

Part (iv) holds similarly since

$$\begin{aligned}
& \left\| \frac{\hat{\beta}_K' (\rho - I_n) Z_{-1}^{0\prime} y \hat{\beta}_K - \beta_K^{*\prime} (\rho - I_n) Z_{-1}^{0\prime} y \beta_K^*}{n\sqrt{n}T} \right\| \\
\leq \ & \left\| \frac{\left(\hat{\beta}_K - \beta_K^*\right)' (\rho - I_n) Z_{-1}^{0\prime} y \hat{\beta}_K}{n\sqrt{n}T} \right\| + \left\| \frac{\beta_K^{*\prime} (\rho - I_n) Z_{-1}^{0\prime} y \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T} \right\| \\
\leq \ & \frac{1}{\sqrt{nT}} \frac{\left\|Z_{-1}^0\right\|}{\sqrt{n}T} \frac{\|y\|}{\sqrt{nT}} \left(\left\| \left(\hat{\beta}_K - \beta_K^*\right) \Theta \right\| \frac{\left\|\hat{\beta}_K\right\|}{\sqrt{n}} - \left\|\hat{\beta}_K - \beta_K^*\right\| \frac{\|\beta_K^* \Theta\|}{\sqrt{n}} \right) \\
= \ & O_p\left(\frac{1}{\sqrt{nT}}\right) = o_p(1).
\end{aligned}$$

For Part (v), notice that

$$\begin{aligned}
& \left\| \frac{\hat{\beta}_K' Z_T^0 Z_T^{0\prime} \hat{\beta}_K - \beta_K^{*\prime} Z_T^0 Z_T^{0\prime} \beta_K^*}{n\sqrt{n}T} \right\| \\
\leq \ & \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' Z_T^0 \right\| \left\| Z_T^{0\prime} \hat{\beta}_K \right\|}{n\sqrt{n}T} + \frac{\left\|\beta_K^{*\prime} Z_T^0\right\| \left\| Z_T^{0\prime} \left(\hat{\beta}_K - \beta_K^*\right) \right\|}{n\sqrt{n}T} \\
\leq \ & \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' Z_T^0 \right\|^2}{n\sqrt{n}T} + 2 \frac{\left\|\beta_K^{*\prime} Z_T^0\right\| \left\| Z_T^{0\prime} \left(\hat{\beta}_K - \beta_K^*\right) \right\|}{n\sqrt{n}T} \\
= \ & \left\| \frac{Z_T^{0\prime} \left(\hat{\beta}_K - \beta_K^*\right)}{\sqrt{n}\sqrt{T}} \right\| \left\{ \frac{1}{\sqrt{n}} \left\| \frac{\left(\hat{\beta}_K - \beta_K^*\right)' Z_T^0}{\sqrt{n}\sqrt{T}} \right\| + 2 \|H_K\| \left\| \frac{\beta^{0\prime} Z_T^0}{n\sqrt{T}} \right\| \right\}. \qquad (54)
\end{aligned}$$

In what follows we show that

$$(\mathrm{v}_1) \ \left\| \frac{\left(\hat{\beta}_K - \beta_K^*\right)' Z_T^0}{\sqrt{n}\sqrt{T}} \right\| = o_p(1) \ \text{ and } (\mathrm{v}_2) \ \left\| \frac{\beta^{0\prime} Z_T^0}{n\sqrt{T}} \right\| = O_p(1).$$

Then, the result of Part (v) follows.

First, Part (v_2) follows because

$$\begin{aligned}
\left\| \frac{\beta^{0\prime} Z_T^0}{n\sqrt{T}} \right\| & \leq \left\| \frac{\beta^{0\prime} Z_T^0(0)}{n\sqrt{T}} \right\| + \left\| \frac{\beta^{0\prime} \left(Z_T^0 - Z_T^0(0)\right)}{n\sqrt{T}} \right\| \\
& \leq \left\| \frac{\beta^{0\prime} Z_T^0(0)}{n\sqrt{T}} \right\| + \frac{1}{\sqrt{n}} \frac{\left\|\beta^0\right\|}{\sqrt{n}} \frac{\left\|Z_T^0 - Z_T^0(0)\right\|}{\sqrt{T}} \\
& = O_p(1) + O_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

where the $O_p\left(\frac{1}{\sqrt{n}}\right)$ term holds by similar arguments used in Lemma 11(a), and the $O_p(1)$

holds since

$$\begin{aligned}
\left\| \frac{\beta^{0\prime} Z_T^0(0)}{n\sqrt{T}} \right\| &\leq \left\| \frac{\beta^{0\prime}\beta^0 F_T}{n\sqrt{T}} \right\| + \left\| \frac{\beta^{0\prime} E_T}{n\sqrt{T}} \right\| \\
&= \left\| \frac{\beta^{0\prime}\beta^0}{n} \right\| \left\| \frac{F_T}{\sqrt{T}} \right\| + \frac{1}{\sqrt{n}} \left\| \frac{\beta^{0\prime} e' l_T}{\sqrt{n}\sqrt{T}} \right\| \\
&= O(1) O_p(1) + \frac{1}{\sqrt{n}} O_p(1) = O_p(1) \qquad (55)
\end{aligned}$$

by Assumptions 6, 3, and Lemma 9(i). Next, for Part (v_2), by the triangle inequality,

$$\begin{aligned}
&\frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' Z_T^0 \right\|}{\sqrt{nT}} \\
\leq\ & \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' Z_T^0(0) \right\|}{\sqrt{nT}} + \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' \left(Z_T^0 - Z_T^0(0)\right) \right\|}{\sqrt{nT}} \\
\leq\ & \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' \beta^0 F_T \right\|}{\sqrt{nT}} + \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' E_T \right\|}{\sqrt{nT}} + \frac{\left\| \left(\hat{\beta}_K - \beta_K^*\right)' \left(Z_T^0 - Z_T^0(0)\right) \right\|}{\sqrt{nT}} \\
=\ & V_a + V_b + V_c, \text{ say.}
\end{aligned}$$

In view of (**??**), (**??**), and by the triangle inequality, we have

$$\begin{aligned}
V_a &\leq \frac{\left\| \bar{\beta}_K' e' f^0 \beta^{0\prime}\beta^0 F_T \right\|}{n\sqrt{nT}\sqrt{T}} + \frac{\left\| \bar{\beta}_K' \beta^0 f^{0\prime} e \beta^0 F_T \right\|}{n\sqrt{nT}\sqrt{T}} + \frac{\left\| \bar{\beta}_K' e' e \beta^0 F_T \right\|}{n\sqrt{nT}\sqrt{T}} \\
&\quad + \sum_{k=1}^{7} \sqrt{n} \left\| \mathcal{R}_k \right\| \frac{\left\| \bar{\beta}_K \right\|}{\sqrt{n}} \frac{\left\| \beta^0 \right\|}{\sqrt{n}} \frac{\left\| F_T \right\|}{\sqrt{T}} \\
&= 2\sqrt{\frac{n}{T}} \left\| \frac{\bar{\beta}_K}{\sqrt{n}} \right\| \left\| \frac{e' f^0}{\sqrt{nT}} \right\| \left\| \frac{\beta^0}{\sqrt{n}} \right\|^2 \left\| \frac{F_T}{\sqrt{T}} \right\| + \left\| \frac{\bar{\beta}_K' e' e \beta_K^*}{n\sqrt{nT}} \right\| \left\| H_K^{-1} \right\| \left\| \frac{F_T}{\sqrt{T}} \right\| \\
&\quad + \sum_{k=1}^{7} \sqrt{n} \left\| \mathcal{R}_k \right\| \frac{\left\| \bar{\beta}_K \right\|}{\sqrt{n}} \frac{\left\| \beta^0 \right\|}{\sqrt{n}} \frac{\left\| F_T \right\|}{\sqrt{T}} \\
&= O_p\left(\sqrt{\frac{n}{T}}\right) + o_p(1) + O_p\left(\sqrt{\frac{n}{T}} \max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) = o_p(1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
V_b &\leq \left\| \frac{\bar{\beta}_K' e' f^0 \beta^{0\prime} E_T}{n\sqrt{nT}\sqrt{T}} \right\| + \left\| \frac{\bar{\beta}_K' \beta^0 f^{0\prime} e E_T}{n\sqrt{nT}\sqrt{T}} \right\| + \left\| \frac{\bar{\beta}_K' e' e E_T}{n\sqrt{nT}\sqrt{T}} \right\| \\
&\quad + \sum_{k=1}^{7} \sqrt{n} \left\| \mathcal{R}_k \right\| \frac{\left\| \bar{\beta}_K \right\|}{\sqrt{n}} \frac{\left\| E_T \right\|}{\sqrt{nT}} \\
&\leq 2\sqrt{\frac{n}{T}} \frac{\left\| \bar{\beta}_K \right\|}{\sqrt{n}} \frac{\left\| e' f^0 \right\|}{\sqrt{nT}} \frac{\left\| \beta^0 \right\|}{\sqrt{n}} \frac{\left\| E_T \right\|}{\sqrt{nT}} + \left\| \frac{\bar{\beta}_K' e' e E_T}{n\sqrt{nT}\sqrt{T}} \right\| + \sum_{k=1}^{7} \sqrt{n} \left\| \mathcal{R}_k \right\| \frac{\left\| \bar{\beta}_K \right\|}{\sqrt{n}} \frac{\left\| E_T \right\|}{\sqrt{nT}} \\
&= \sqrt{\frac{n}{T}} O_p(1) + \frac{\left\| \bar{\beta}_K' e' e E_T \right\|}{n\sqrt{nT}\sqrt{T}} + \sqrt{\frac{n}{T}} O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right),
\end{aligned}$$

where the last line holds by the same principle of the last line of Part V_1 with $\frac{\|E_T\|}{\sqrt{nT}} = \frac{\|e'l_T\|}{\sqrt{nT}} = O_p(1)$ by Lemma 9 (i) and (i*). For the remaining term $\frac{\|\bar{\beta}'_K e'eE_T\|}{n\sqrt{n}T\sqrt{T}}$, using $\hat{\beta}_K = \bar{\beta}_K \Lambda_{nT,K}$ and $\beta^*_K = \beta^0 H_K$ and by the triangle inequality,

$$\frac{\left\|\bar{\beta}'_K e'eE_T\right\|}{n\sqrt{n}T\sqrt{T}} \leq \left\|\Lambda^{-1}_{nT,K}\right\| \frac{\left\|(\hat{\beta}_K - \beta^*_K)'e'eE_T\right\|}{n\sqrt{n}T\sqrt{T}} + \left\|\Lambda^{-1}_{nT,K}\right\| \left\|H^{-1}_K\right\| \frac{\left\|\beta^{0\prime}e'eE_T\right\|}{n\sqrt{n}T\sqrt{T}}.$$

The first term is

$$\begin{aligned}
\left\|\Lambda^{-1}_{nT,K}\right\| \frac{\left\|(\hat{\beta}_K - \beta^*_K)'e'eE_T\right\|}{n\sqrt{n}T\sqrt{T}} &\leq \left\|\Lambda^{-1}_{nT,K}\right\| \left\|\hat{\beta}_K - \beta^*_K\right\| \frac{\|e'e\|}{nT} \frac{\|E_T\|}{\sqrt{nT}} \\
&= O_p(1)\, O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) O_p(1) = o_p(1),
\end{aligned}$$

where the last line holds by Lemma 15(b), Lemma 9(e). Also the second term is

$$\left\|\Lambda^{-1}_{nT,K}\right\| \left\|H^{-1}_K\right\| \frac{\left\|\beta^{0\prime}e'eE_T\right\|}{n\sqrt{n}T\sqrt{T}} = O_p(1)\, O_p(1)\, o_p(1) = o_p(1)$$

by Lemma 9(h). Therefore, $\frac{\|\bar{\beta}'_K e'eE_T\|}{n\sqrt{n}T\sqrt{T}} = o_p(1)$ and so in consequence,

$$V_b = o_p(1). \tag{56}$$

Before we start the proof of $V_c = o_p(1)$, we define

$$F^*_T = \sum_{t=1}^{T} \left(1 - \frac{t}{T}\right) f_t$$

and

$$E^*_T = \sum_{t=1}^{T} \left(1 - \frac{t}{T}\right) e_t,$$

where $e_t = (e_{1t}, ..., e_{nt})'$. Using this notation, we write

$$Z^0_T - Z^0_T(0) \sim -\frac{1}{\sqrt{n}} \Theta \beta^0 F^*_T - \frac{1}{\sqrt{n}} \Theta E^*_T.$$

Since $\frac{E\|F^*_T\|^2}{T}, \frac{E\|\Theta E^*_T\|^2}{nT} < M$, we have

$$\frac{\|F^*_T\|}{\sqrt{T}} = O_p(1)$$

and

$$\frac{\|\Theta E^*_T\|}{\sqrt{nT}} = O_p(1).$$

By the triangle inequality,

$$V_c \leq \frac{\left\|\left(\hat{\beta}_K - \beta^*_K\right)' \Theta \beta^0 F^*_T\right\|}{n\sqrt{T}} + \frac{\left\|\left(\hat{\beta}_K - \beta^*_K\right)' \Theta E^*_T\right\|}{n\sqrt{T}} = V_{ca} + V_{cb}, \text{ say.}$$

Similar to V_a, we have

$$\begin{aligned}
V_{ca} &\leq \frac{\left\|\bar{\beta}_K' e' f^0 \beta^{0\prime} \Theta \beta^0 F_T^*\right\|}{n^2 T\sqrt{T}} + \frac{\left\|\bar{\beta}_K' \beta^0 f^{0\prime} e \Theta \beta^0 F_T^*\right\|}{n^2 T\sqrt{T}} + \frac{\left\|\bar{\beta}_K' e' e \Theta \beta^0 F_T^*\right\|}{n^2 T\sqrt{T}} \\
&\quad + \sum_{k=1}^{7} \|\mathcal{R}_k\| \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\left\|\Theta\beta^0\right\|}{\sqrt{n}} \frac{\|F_T^*\|}{\sqrt{T}} \\
&\leq \frac{2}{\sqrt{T}} \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\left\|e' f^0\right\|}{\sqrt{nT}} \frac{\left\|\beta^0\right\|}{\sqrt{n}} \frac{\left\|\Theta\beta^0\right\|}{\sqrt{n}} \frac{\|F_T^*\|}{\sqrt{T}} + \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\|e' e\|}{nT} \frac{\left\|\Theta\beta^0\right\|}{\sqrt{n}} \frac{\|F_T^*\|}{\sqrt{T}} \\
&\quad + \sum_{k=1}^{7} \|\mathcal{R}_k\| \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\left\|\Theta\beta^0\right\|}{\sqrt{n}} \frac{\|F_T^*\|}{\sqrt{T}} \\
&= O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) + \frac{1}{\sqrt{T}} O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) = o_p(1).
\end{aligned}$$

Similar to V_b, we have

$$\begin{aligned}
V_{cb} &\leq \left\|\frac{\bar{\beta}_K' e' f^0 \beta^{0\prime} \Theta E_T^*}{n^2 T\sqrt{T}}\right\| + \left\|\frac{\bar{\beta}_K' \beta^0 f^{0\prime} e \Theta E_T^*}{n^2 T\sqrt{T}}\right\| + \left\|\frac{\bar{\beta}_K' e' e \Theta E_T^*}{n^2 T\sqrt{T}}\right\| \\
&\quad + \sum_{k=1}^{7} \|\mathcal{R}_k\| \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\|\Theta E_T^*\|}{\sqrt{nT}} \\
&\leq \frac{2}{\sqrt{T}} \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\left\|e' f^0\right\|}{\sqrt{nT}} \frac{\left\|\beta^0\right\|}{\sqrt{n}} \frac{\|\Theta E_T^*\|}{\sqrt{nT}} + \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\|e' e\|}{nT} \frac{\|\Theta E_T^*\|}{\sqrt{nT}} + \sum_{k=1}^{7} \|\mathcal{R}_k\| \frac{\left\|\bar{\beta}_K\right\|}{\sqrt{n}} \frac{\|\Theta E_T^*\|}{\sqrt{nT}} \\
&= \frac{1}{\sqrt{T}} O_p(1) + O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) + \sqrt{\frac{1}{T}} O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) = o_p(1).
\end{aligned}$$

>From V_{ca} and V_{cb}, we have the required result that $V_c = o_p(1)$. Combining V_a, V_b, and V_c we have the required result for Part (v_2), and in consequence, we complete the proof of Part (v).

Finally, for Part (vi), notice by the triangle inequality that

$$\begin{aligned}
&\left\|\frac{\hat{\beta}_K' y' y \hat{\beta} - \beta_K^{*\prime} y' y \beta_K^*}{n\sqrt{n}T}\right\| = \left\|\frac{\left(\hat{\beta}_K - \beta_K^*\right)' y' y \hat{\beta}_K + \beta_K^{*\prime} y' y \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T}\right\| \\
&= \left\|\frac{\left(\hat{\beta}_K - \beta_K^*\right)' y' y \left(\hat{\beta}_K - \beta_K^*\right) + \left(\hat{\beta}_K - \beta_K^*\right)' y' y \beta_K^* + \beta_K^{*\prime} y' y \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T}\right\| \\
&\leq \left\|\frac{\left(\hat{\beta}_K - \beta_K^*\right)' y' y \left(\hat{\beta}_K - \beta_K^*\right)}{n\sqrt{n}T}\right\| + 2\left\|\frac{\left(\hat{\beta}_K - \beta_K^*\right)' y' y \beta_K^*}{n\sqrt{n}T}\right\| \\
&\leq \frac{1}{\sqrt{n}} \left\|\hat{\beta}_K - \beta_K^*\right\|^2 \frac{\|y\|^2}{nT} + 2\left\|\frac{\left(\hat{\beta}_K - \beta_K^*\right)' y' y \beta^0}{n\sqrt{n}T}\right\| \|H_K\| \\
&= VI_a + VI_b, \text{ say.}
\end{aligned}$$

By Lemma 15(b) and Lemma 9(k), the first term

$$VI_a = \frac{1}{\sqrt{n}} O_p(1) = o_p(1).$$

Next we consider the term VI_b. Using (??), we expand $\left\| \frac{(\hat{\beta}_K - \beta_K^*)' y' y \beta^0}{n\sqrt{n}T} \right\|$. Using $\frac{\|y'y\|}{nT} \leq \frac{\|y\|^2}{nT} = O_p(1)$ by Lemma 9(k), $\left\| \frac{\beta^0}{\sqrt{n}} \right\| = O(1)$, and (??), it is not difficult to find that all the expanded terms of $\left\| \frac{(\hat{\beta}_K - \beta_K^*)' y' y \beta^0}{n\sqrt{n}T} \right\|$ are of order $O_p\left(\sqrt{\frac{n}{T}}\right) = o_p(1)$ (under Assumption 10) except for the term $\left\| \frac{\bar{\beta}_K' e' e y' y \beta^0}{n^2\sqrt{n}T^2} \right\|$. Expand $\frac{\bar{\beta}_K' e' e y' y \beta^0}{n^2\sqrt{n}T^2}$ by substituting $f^0\beta^{0\prime} + e$ for y and apply the triangle inequality. Then,

$$\begin{aligned}
& \left\| \frac{\bar{\beta}_K' e' e y' y \beta^0}{n^2\sqrt{n}T^2} \right\| \\
\leq & \left\| \frac{\bar{\beta}_K' e' e \beta^0 f^{0\prime} f^0 \beta^{0\prime} \beta^0}{n^2\sqrt{n}T^2} \right\| + \left\| \frac{\bar{\beta}_K' e' e \beta^0 f^{0\prime} e \beta^0}{n^2\sqrt{n}T^2} \right\| + \left\| \frac{\bar{\beta}_K' e' e e' f^0 \beta^{0\prime} \beta^0}{n^2\sqrt{n}T^2} \right\| + \left\| \frac{\bar{\beta}_K' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\| \\
\leq & \frac{\left\| \bar{\beta}_K' e' e \beta_K^* \right\|}{n\sqrt{n}T} \left\| H_K^{-1} \right\| \left\| \frac{f^{0\prime} f^0}{T} \right\| \left\| \frac{\beta^{0\prime} \beta^0}{n} \right\| \\
& + \frac{2}{\sqrt{T}} \left\| \frac{\bar{\beta}_K}{\sqrt{n}} \right\| \left\| \frac{\beta^0}{\sqrt{n}} \right\|^2 \left\| \frac{e' e}{nT} \right\| \left\| \frac{f^{0\prime} e}{\sqrt{n}\sqrt{T}} \right\| + \left\| \frac{\bar{\beta}_K' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\| \\
= & o_p(1) O_p(1) O_p(1) + \frac{1}{\sqrt{T}} O_p(1) O(1) O_p\left(\max\left\{ \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}} \right\} \right) O_p(1) + \left\| \frac{\bar{\beta}_K' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\|,
\end{aligned}$$

where the last line holds by Equation (??), Lemma 14, Assumption 7, Lemmas 9(e) and (f). For the remaining term $\left\| \frac{\bar{\beta}_K' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\|$, using $\hat{\beta}_K = \bar{\beta}_K \Lambda_{nT,K}$ and $\beta_K^* = \beta^0 H_K$ and the triangle inequality, we have

$$\begin{aligned}
& \left\| \frac{\bar{\beta}_K' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\| = \left\| \Lambda_{nT,K}^{-1} \frac{\hat{\beta}_K' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\| \\
\leq & \left\| \Lambda_{nT,K}^{-1} \right\| \left\| \frac{\left(\hat{\beta}_K - \beta_K^* \right)' e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\| + \left\| \Lambda_{nT,K}^{-1} \right\| \left\| H_K \right\| \left\| \frac{\beta^{0\prime} e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right\| \\
\leq & \left\| \Lambda_{nT,K}^{-1} \right\| \left\| \hat{\beta}_K - \beta_K^* \right\| \left\| \frac{e' e}{nT} \right\|^2 \left\| \frac{\beta^0}{\sqrt{n}} \right\| + \left\| \Lambda_{nT,K}^{-1} \right\| \left\| H_K \right\| tr\left(\frac{\beta^{0\prime} e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right) \\
= & O_p(1) O_p(1) O_p\left(\max\left\{ \frac{1}{n}, \frac{1}{T} \right\} \right) O(1) + O_p(1) O_p(1) O_p\left(\frac{1}{\sqrt{n}} \right) = o_p(1),
\end{aligned}$$

where the last line holds by Lemmas 13, 15(b), 9(e), and 14, by Assumption 6, and by

$$tr\left(\frac{\beta^{0\prime} e' e e' e \beta^0}{n^2\sqrt{n}T^2} \right) = tr\left(\frac{e e' e \beta^0 \beta^{0\prime} e'}{n^2\sqrt{n}T^2} \right) \leq \frac{1}{\sqrt{n}} \frac{\|e\|^2}{nT} \frac{tr\left(\beta^{0\prime} e' e \beta^0 \right)}{nT} = \frac{1}{\sqrt{n}} O_p(1) O_p(1)$$

due to $\sup_{it} E\left(e_{it}^2\right) < M$ and Lemma 9(g). From this, we deduce that $\left\| \frac{\bar{\beta}_K' e' e y' y \beta^0}{n^2\sqrt{n}T^2} \right\| = o_p(1)$, which leads to

$$VI_b = o_p(1).$$

Therefore,

$$\left\| \frac{\hat{\beta}_K' y' y \hat{\beta}_K - \beta_K^{*\prime} y' y \beta_K^*}{n\sqrt{n}T} \right\| = o_p(1),$$

as required, and this completes the proof part (vi). ■

■

2.3 Appendix C: Estimation of the Number of Factors

Proof of Theorem 3

The proof is similar to the proof of Theorem 2 of Bai and Ng (2002). Thus, we only sketch the proof here.

Proof of Part (a).

First, notice that the required result follows if we show that for $r \neq K$,

$$P\{PC(r) < PC(K)\} \to 0$$

as $(n, T \to \infty)$ with $\frac{n}{T} \to 0$. By definition,

$$P\{PC(r) < PC(K)\} = P\left\{W_{nT}\left(\hat{\beta}_K, K\right) - W_{nT}\left(\hat{\beta}_r, r\right) > (r-K)\, G_{nT}\right\}.$$

Now we consider two cases, (i) $1 \leq r < K$ and (ii) $K < r < \bar{K}$.

Case (i): $1 \leq r < K$.

Notice that

$$\begin{aligned}
& W_{nT}\left(\hat{\beta}_K, K\right) - W_{nT}\left(\hat{\beta}_r, r\right) \\
= \; & W_{nT}\left(\hat{\beta}_K, K\right) - W_{nT}\left(\beta_K^*, K\right) + W_{nT}\left(\beta_K^*, K\right) - W_{nT}\left(\beta_r^*, r\right) \\
& + W_{nT}\left(\beta_r^*, r\right) - W_{nT}\left(\hat{\beta}_r, r\right) \\
= \; & I_1 + II_1 + III_1, \text{ say.}
\end{aligned}$$

where β_K^* is the rotated matrix of factor loadings defined in lemma 15. In what follows, we show that I_1 and III_1 converge in probability to zero and II_1 converges in probability to a negative constant. Then, since $(r-K)\, G_{nT} \to 0$,

$$P\left\{W_{nT}\left(\hat{\beta}_K, K\right) - W_{nT}\left(\hat{\beta}_r, r\right) > (r-K)\, G_{nT}\right\} \to 0,$$

as required.

For I_1, notice that for $1 \leq r \leq K$,

$$\begin{aligned}
\left| W_{nT}\left(\beta_r^*, r\right) - W_{nT}\left(\hat{\beta}_r, r\right)\right| &= \left| \frac{tr\left(\hat{y}\left(P_{\hat{\beta}_r} - P_{\beta_r^*}\right)\hat{y}'\right)}{nT}\right| \leq \left\| P_{\hat{\beta}_r} - P_{\beta_r^*}\right\| \left\| \frac{\hat{y}'\hat{y}}{nT}\right\| \\
&= o_p(1)\, O_p(1) = o_p(1),
\end{aligned}$$

where the last equality holds by modified arguments used in the proof of Lemma 3 with $\hat{\beta}_r$ and β_r^* and since $\left\|\frac{\hat{y}'\hat{y}}{nT}\right\| = O_p(1)$. So, we have

$$I_1,\ III_1 \to_p 0.$$

Next, for II_1 notice that

$$II_1 = W_{nT}\left(\beta_K^*, K\right) - W_{nT}\left(\beta_r^*, r\right) = W_{nT}\left(\beta^0, K\right) - W_{nT}\left(\beta_r^*, r\right) = \frac{tr\left(\hat{y}\left(P_{\beta_r^*} - P_{\beta^0}\right)\hat{y}'\right)}{nT}$$

since H_K, the rotating matrix defined in lemma 15, is asymptotically full rank $K \times K$ matrix. Also, since

$$\begin{aligned} & \frac{tr\left(\hat{y}\left(P_{\beta_r^*} - P_{\beta^0}\right)\hat{y}'\right)}{nT} - \frac{tr\left(y\left(P_{\beta_r^*} - P_{\beta^0}\right)y'\right)}{nT} \\ = \; & tr\left(\left(P_{\beta_r^*} - P_{\beta^0}\right)\left(\frac{\hat{y}'\hat{y}}{nT} - \frac{y'y}{nT}\right)\right) \leq \left\|P_{\beta_r^*} - P_{\beta^0}\right\| \left\|\frac{\hat{y}'\hat{y}}{nT} - \frac{y'y}{nT}\right\| \\ = \; & \left(\left\|P_{\beta_r^*}\right\| + \left\|P_{\beta^0}\right\|\right)\left\|\frac{\hat{y}'\hat{y}}{nT} - \frac{y'y}{nT}\right\| = O_p(1)\, o_p(1) = o_p(1), \end{aligned}$$

the required result that II_1 converges in probability to a negative constant follows if we show that $\frac{tr\left(y\left(P_{\beta_r^*} - P_{\beta^0}\right)y'\right)}{nT}$ converges to a negative constant. For this, using the definition of $y = f^0\beta^{0\prime} + e$, we may write

$$\begin{aligned} & \frac{tr\left(y\left(P_{\beta_r^*} - P_{\beta^0}\right)y'\right)}{nT} \\ = \; & \frac{tr\left(f^0\beta^{0\prime}\left(P_{\beta_r^*} - P_{\beta^0}\right)\beta^0 f^{0\prime}\right)}{nT} + 2\frac{tr\left(f^0\beta^{0\prime}\left(P_{\beta_r^*} - P_{\beta^0}\right)e'\right)}{nT} + \frac{tr\left(e\left(P_{\beta_r^*} - P_{\beta^0}\right)e'\right)}{nT} \\ = \; & II_{11} + 2II_{12} + II_{13}, \text{ say.} \end{aligned}$$

Notice that

$$\begin{aligned} |II_{12}| &= \left|\frac{tr\left(\left(P_{\beta_r^*} - P_{\beta^0}\right)e'f^0\beta^{0\prime}\right)}{nT}\right| \leq \left\|P_{\beta_r^*} - P_{\beta^0}\right\| \left\|\frac{e'f^0\beta^{0\prime}}{nT}\right\| \\ &\leq \frac{1}{\sqrt{T}}\left(\left\|P_{\beta_r^*}\right\| + \left\|P_{\beta^0}\right\|\right)\left\|\frac{e'f^0}{\sqrt{nT}}\right\|\left\|\frac{\beta^0}{\sqrt{n}}\right\| = \frac{1}{\sqrt{T}} O_p(1)\, O_p(1)\, O(1) = o_p(1), \end{aligned}$$

where the first equality of the second line holds by Lemma 9(f). Also,

$$\begin{aligned} |II_{13}| &= \left|\frac{tr\left(\left(P_{\beta_r^*} - P_{\beta^0}\right)e'e\right)}{nT}\right| \leq \left\|P_{\beta_r^*} - P_{\beta^0}\right\| \left\|\frac{e'e}{nT}\right\| \\ &\leq \left(\left\|P_{\beta_r^*}\right\| + \left\|P_{\beta^0}\right\|\right)\left\|\frac{e'e}{nT}\right\| = O_p(1)\, O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) = o_p(1), \end{aligned}$$

where the first equality of the second line holds by Lemma 9(e).

The convergence of II_{11} to a negative number can be obtained by following the same lines of arguments in Bai and Ng (2002, proof of Lemma 3) or in Stock and Watson (1998, proof of Theorem 2 on page 48), and we omit the details here. Thus, $\frac{tr\left(y\left(P_{\beta_r^*} - P_{\beta^0}\right)y'\right)}{nT}$ converges to a negative constant, as required.

Case (ii): $K + 1 \leq r \leq \bar{K}$.

We follow similar arguments in the proof of Lemma 4 of Bai and Ng (2002). For $K < r \leq \bar{K}$, we have

$$\begin{aligned} \left|W_{nT}\left(\hat{\beta}_r, r\right) - W_{nT}\left(\hat{\beta}_K, K\right)\right| &\leq \left|W_{nT}\left(\hat{\beta}_r, r\right) - W_{nT}\left(\beta^0, K\right)\right| + \left|W_{nT}\left(\beta^0, K\right) - W_{nT}\left(\hat{\beta}_K, K\right)\right| \\ &\leq 2 \max_{r \in \left\{K, K+1, \ldots, \bar{K}\right\}} \left|W_{nT}\left(\hat{\beta}_r, r\right) - W_{nT}\left(\beta^0, K\right)\right|. \end{aligned}$$

Since the set $\left\{K, K+1, ..., \bar{K}\right\}$ is finite, we fix r and show that $\left|W_{nT}\left(\hat{\beta}_r, r\right) - W_{nT}\left(\beta^0, K\right)\right| = O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right)$. Then, since $\min\{n, T\}(r - K)\, G_{nT} \to \infty$ if $K+1 \leq r \leq \bar{K}$, as assumed in the theorem, we have the required result that $P\left\{W_{nT}\left(\hat{\beta}_K, K\right) - W_{nT}\left(\hat{\beta}_r, r\right) > (r-K)\, G_{nT}\right\} \to 0$.

Let H_r^+ be the generalized inverse of H_r, *i.e.*, $H_r H_r^+ = I_r$. By definition, we have

$$\begin{aligned}
\hat{y} &= f^0 H_r^{+\prime} H_r' \beta^{0\prime} + e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \\
&= f^0 H_r^{+\prime} \hat{\beta}_r' + f^0 H_r^{+\prime} \left(\beta_r^* - \hat{\beta}_r\right)' + e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right).
\end{aligned}$$

Then,

$$\begin{aligned}
& W_{nT}\left(\beta^0, K\right) - W_{nT}\left(\hat{\beta}_r, r\right) \\
=\ & \frac{tr\left(\hat{y} Q_{\beta^0} \hat{y}'\right)}{nT} - \frac{tr\left(\hat{y} Q_{\hat{\beta}_r} \hat{y}'\right)}{nT} \\
=\ & \frac{1}{nT} tr \left\{ \begin{array}{c} \left(e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right) Q_{\beta^0} \left(e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right)' \\ - \left(\begin{array}{c} f^0 H_r^{+\prime} \left(\beta_r^* - \hat{\beta}_r\right)' + \\ e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \end{array} \right) Q_{\hat{\beta}_r} \left(\begin{array}{c} f^0 H_r^{+\prime} \left(\beta_r^* - \hat{\beta}_r\right)' + \\ e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \end{array} \right)' \end{array} \right. \\
=\ & \frac{1}{nT} tr \left\{ e \left(P_{\hat{\beta}_r} - P_{\beta^0}\right) e' \right\} \\
& + \frac{1}{nT} tr \left\{ \left(\begin{array}{c} \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' \\ + Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \end{array} \right) \left(P_{\hat{\beta}_r} - P_{\beta^0}\right) \left(\begin{array}{c} \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' \\ + Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \end{array} \right)' \right\} \\
& - 2 \frac{1}{nT} tr \left\{ e \left(P_{\hat{\beta}_r} - P_{\beta^0}\right) \left(\left(\hat{\rho}_{pool} - 1\right) l_T \alpha' + Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right)' \right\} \\
& - \frac{1}{nT} tr \left\{ f^0 H_r^{+\prime} \left(\beta_r^* - \hat{\beta}_r\right)' Q_{\hat{\beta}_r} \left(\beta_r^* - \hat{\beta}_r\right) H_r^+ f^{0\prime} \right\} \\
& - 2 \frac{1}{nT} tr \left\{ \left(e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right) Q_{\hat{\beta}_r} \left(\beta_r^* - \hat{\beta}_r\right) H_r^+ f^{0\prime} \right\} \\
=\ & I_2 + II_2 + 2III_2 + IV_2 + 2V_2, \text{ say.}
\end{aligned}$$

First, for II_2, since

$$\begin{aligned}
II_2 &= \frac{\left| tr \left\{ \left(P_{\hat{\beta}_r} - P_{\beta^0}\right) \left(\begin{array}{c} \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' \\ + Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \end{array} \right) \left(\begin{array}{c} \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' \\ + Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \end{array} \right)' \right\} \right|}{nT} \\
&\le \left\| P_{\hat{\beta}_r} - P_{\beta^0} \right\| \frac{\left\| \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' + Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right) \right\|^2}{nT} \\
&\le \frac{2}{T} \left\| P_{\hat{\beta}_r} - P_{\beta^0} \right\| \left\{ \begin{array}{c} T^2 \left(\hat{\rho}_{pool} - 1\right)^2 \frac{\left\| l_T \alpha' \right\|^2}{nT^2} \\ + \left(T^2 \left(\hat{\rho}_{pool} - 1\right)^2 + T^2 \left\| \rho - I_n \right\|^2\right) \frac{\left\| Z_{-1}^0 \right\|^2}{nT^2} \end{array} \right\} \\
&= \frac{1}{T} O_p(1) \left(O_p(1)\, O_p\left(\frac{1}{T}\right) + O_p(1)\, O_p(1) \right)
\end{aligned}$$

we have

$$II_2 \le O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right),$$

as required. Next, for III_2, notice that

$$III_2 \leq \left| \frac{tr\left\{\left(\left(\hat{\rho}_{pool}-1\right)l_T\alpha' + Z_{-1}^0\left(\hat{\rho}_{pool}I_n-\rho\right)\right)\left(P_{\hat{\beta}_r}-P_{\beta^0}\right)e'\right\}}{nT}\right|$$

$$\begin{aligned}
&\leq \left|\frac{tr\left\{\left(\hat{\rho}_{pool}-1\right)\left(P_{\hat{\beta}_r}-P_{\beta^0}\right)e'l_T\alpha'\right\}}{nT}\right| \\
&\quad +\frac{1}{2}\left|\frac{tr\left\{\left(P_{\hat{\beta}_r}-P_{\beta^0}\right)\left(e'Z_{-1}^0\left(\hat{\rho}_{pool}I_n-\rho\right)\right)+\left(\hat{\rho}_{pool}I_n-\rho\right)Z_{-1}^{0\prime}e\right\}}{nT}\right| \\
&\leq \left\|P_{\hat{\beta}_r}-P_{\beta^0}\right\|\left\{\begin{array}{c} \left|T\left(\hat{\rho}_{pool}-1\right)\right|\left\|\frac{e'l_T\alpha'}{nT^2}\right\| + \frac{1}{2}\left|T\left(\hat{\rho}_{pool}-1\right)\right| \frac{\left\|e'Z_{-1}^0+Z_{-1}^{0\prime}e\right\|}{nT^2} \\ \frac{1}{2}\frac{\left\|e'Z_{-1}^0(\rho-I_n)+(\rho-I_n)Z_{-1}^{0\prime}e\right\|}{nT}\end{array}\right\} \\
&= O_p(1)\left(III_{21}+\frac{1}{2}III_{22}+\frac{1}{2}III_{23}\right), \text{ say.}
\end{aligned}$$

Since

$$III_{21} \leq \frac{1}{T\sqrt{T}}\frac{\|e'l_T\|}{\sqrt{nT}}\frac{\|\alpha\|}{\sqrt{n}} = \frac{1}{T\sqrt{T}}O_p(1)\,O_p(1),$$

we have

$$III_{21} \leq O_p\left\{\max\left\{\frac{1}{n},\frac{1}{T}\right\}\right\}.$$

Next,

$$III_{22}+III_{23} = \frac{1}{T}O_p(1) \leq O_p\left(\max\left\{\frac{1}{n},\frac{1}{T}\right\}\right)$$

because $\frac{\left\|e'Z_{-1}^0+Z_{-1}^{0\prime}e\right\|}{nT} = O_p(1)$ and $\frac{\left\|e'Z_{-1}^0(\rho-I_n)+(\rho-I_n)Z_{-1}^{0\prime}e\right\|}{nT} = O_p\left(\frac{1}{\sqrt{n}T}\right)$.[4] In view of III_{21}, III_{22}, and III_{23}, we have the required result that

$$III_2 = O_p\left(\max\left\{\frac{1}{n},\frac{1}{T}\right\}\right).$$

for IV_2, notice that[5]

$$\begin{aligned}
IV_2 &\leq \left|\frac{tr\left\{Q_{\hat{\beta}_r}\left(\beta_r^*-\hat{\beta}_r\right)H_r^+f^{0\prime}f^0H_r^{+\prime}\left(\beta_r^*-\hat{\beta}_r\right)'\right\}}{nT}\right| \\
&\leq \left\|\frac{\beta_r^*-\hat{\beta}_r}{\sqrt{n}}\right\|^2\left\|H_r^+\right\|^2\left\|\frac{f^{0\prime}f^0}{T}\right\|\left(1+\left\|P_{\hat{\beta}_r}\right\|\right) \\
&= O_p\left(\max\left\{\frac{1}{n},\frac{1}{T}\right\}\right)O_p(1)\,O_p(1)\,O_p(1) = O_p\left(\max\left\{\frac{1}{n},\frac{1}{T}\right\}\right).
\end{aligned}$$

[4] The detailed proof is provided at the end.

[5] The proof of $\frac{\beta_r^*-\hat{\beta}_r}{\sqrt{n}} = \max\left\{\frac{1}{\sqrt{n}},\frac{1}{\sqrt{T}}\right\}$ is similar to that of Lemma 9(a), and we omit it here.

Thus,

$$IV_2 = O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right).$$

Next, for V_2 notice that

$$\begin{aligned}
V_2 &= \left|\frac{tr\left\{\left(e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right) Q_{\hat{\beta}_r} \left(\beta_r^* - \hat{\beta}_r\right) H_r^+ f^{0\prime}\right\}}{nT}\right| \\
&\leq \left|\frac{tr\left\{H_r^+ f^{0\prime} \left(e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right) \left(\beta_r^* - \hat{\beta}_r\right)\right\}}{nT}\right| \\
&\quad + \left|\frac{tr\left\{H_r^+ f^{0\prime} \left(e - \left(\hat{\rho}_{pool} - 1\right) l_T \alpha' - Z_{-1}^0 \left(\hat{\rho}_{pool} I_n - \rho\right)\right) P_{\hat{\beta}_r} \left(\beta_r^* - \hat{\beta}_r\right)\right\}}{nT}\right| \\
&\leq \frac{1}{\sqrt{T}} \left(1 + \left\|P_{\hat{\beta}_r}\right\|\right) \left\|H_r^+\right\| \left(\begin{array}{c} \frac{\left\|f^{0\prime} e\right\|}{\sqrt{nT}} + \frac{1}{\sqrt{T}} \left|T\left(\hat{\rho}_{pool} - 1\right)\right| \frac{\|l_T\|}{\sqrt{T}} \frac{\left\|f^0\right\|}{\sqrt{T}} \frac{\|\alpha\|}{\sqrt{n}} \\ + \frac{\left\|Z_{-1}^0\right\|}{\sqrt{nT}} \frac{\left\|f^0\right\|}{\sqrt{T}} \left(\left|T\left(\hat{\rho}_{pool} - 1\right)\right| + T\left\|\rho - I_n\right\|\right) \end{array}\right) \left\|\frac{\beta_r^* - \hat{\beta}_r}{\sqrt{n}}\right\| \\
&= \frac{1}{\sqrt{T}} O_p(1) O_p(1) \left(O_p(1) + O_p(1) + O_p(1)\right) O_p\left(\max\left\{\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{T}}\right\}\right) \\
&\leq O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right),
\end{aligned}$$

and we have

$$V_2 = O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right),$$

as required.

Notice that for any $\beta_r = \left(\beta^0 \vdots \beta_{r-k}^*\right)$,

$$W_{nT}\left(\hat{\beta}_r, r\right) \leq W_{nT}\left(\beta_r, r\right) \leq W_{nT}\left(\beta^0, K\right).$$

Then, in view of II_2, III_2, IV_2 and V_2, we have

$$0 \leq W_{nT}\left(\beta^0, K\right) - W_{nT}\left(\hat{\beta}_r, r\right) \leq \frac{1}{nT} tr\left\{e\left(P_{\hat{\beta}_r} - P_{\beta^0}\right) e'\right\} + O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right),$$

which yields

$$0 \leq \frac{1}{nT} tr\left(e P_{\beta^0} e'\right) \leq \frac{1}{nT} tr\left(e P_{\hat{\beta}_r} e'\right) \leq O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right).$$

Since

$$\begin{aligned}
\frac{1}{nT} tr\left(e P_{\beta^0} e'\right) &\leq tr\left\{\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1} \frac{\beta^{0\prime} e' e \beta^0}{n^2 T}\right\} \leq \frac{1}{n} \left\|\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\right\| \frac{\left\|e\beta^0\right\|^2}{nT} \\
&= \frac{1}{n} O_p(1) O_p(1),
\end{aligned}$$

we have

$$\frac{1}{nT} tr\left(e P_{\hat{\beta}_r} e'\right) \leq O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right),$$

and so,

$$I_2 \leq O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right).$$

Therefore, we have

$$\left|W_{nT}\left(\hat{\beta}_r, r\right) - W_{nT}\left(\beta^0, K\right)\right| = O_p\left(\max\left\{\frac{1}{n}, \frac{1}{T}\right\}\right),$$

and this completes the proof of Case (ii). ■

Proof of Part (b).
Part (b) follows from the same arguments as those in the proof of Corollary 1 of Bai and Ng (2002). ■

Proof of $\frac{\left\|e' Z^0_{-1} + Z^{0\prime}_{-1} e\right\|}{nT} = O_p(1)$.
For the required result, it is enough to show that

$$\frac{\left\|f^{0\prime} Z^0_{-1}\right\|}{\sqrt{n}T} = O_p(1),$$

because $\frac{\left\|y' Z^0_{-1} + Z^{0\prime}_{-1} y\right\|}{nT} = O_p(1)$ and

$$\frac{\left\|\beta^0 f^{0\prime} Z^0_{-1} + Z^{0\prime}_{-1} f^0 \beta^{0\prime}\right\|}{nT} \leq 2\frac{\left\|\beta^0\right\|}{\sqrt{n}} \frac{\left\|f^{0\prime} Z^0_{-1}\right\|}{\sqrt{n}T} = O(1) \frac{\left\|f^{0\prime} Z^0_{-1}\right\|}{\sqrt{n}T}.$$

Since $z^0_t = \sum_{s=1}^t \rho^{t-s} y_s = \sum_{s=1}^t \rho^{t-s} \beta^0 f_s + \sum_{s=1}^t \rho^{t-s} e_s$,

$$\begin{aligned}
\frac{\left\|f^{0\prime} Z^0_{-1}\right\|}{\sqrt{n}T} &= \frac{1}{\sqrt{n}T}\left\|\sum_{t=2}^T f^0_t z^{0\prime}_{t-1}\right\| \\
&\leq \frac{1}{\sqrt{n}T}\left\|\sum_{t=2}^T f^0_t \left(\sum_{s=1}^{t-1} f^{0\prime}_s \beta^{0\prime} \rho^{t-s}\right)\right\| + \frac{1}{\sqrt{n}T}\left\|\sum_{t=2}^T f^0_t \left(\sum_{s=1}^{t-1} e'_s \rho^{t-s}\right)\right\|.
\end{aligned}$$

Notice that

$$\begin{aligned}
& \frac{1}{nT^2} E\left\|\sum_{t=2}^T f^0_t \left(\sum_{s=1}^{t-1} f^{0\prime}_s \beta^{0\prime} \rho^{t-s}\right)\right\|^2 \\
= & \frac{1}{nT^2} tr \sum_{t=2}^T \sum_{s=1}^{t-1} \sum_{p=2}^T \sum_{q=1}^{p-1} E\left[f^0_t f^{0\prime}_s \beta^{0\prime} \rho^{t-s} \rho^{t-q} \beta^0 f^0_q f^{0\prime}_p\right] \\
= & \frac{1}{nT^2} tr \sum_{t=2}^T \sum_{s=1}^{t-1} \sum_{p=2}^T \sum_{q=1}^{p-1} E\left[\beta^{0\prime} \rho^{t-s} \rho^{t-q} \beta^0 f^0_q f^{0\prime}_p f^0_t f^{0\prime}_s\right] \\
= & \frac{1}{nT^2} tr \sum_{t=2}^T \sum_{s=1}^{t-1} \sum_{p=2}^T \sum_{q=1}^{p-1} \sum_{i=1}^n \beta^0_i \beta^{0\prime}_i E\left[\rho_i^{2t-s-q} f^0_q f^{0\prime}_p f^0_t f^{0\prime}_s\right] \\
= & O(1).
\end{aligned}$$

Next,

$$\begin{aligned}
& \frac{1}{nT^2}E\left\|\sum_{t=2}^{T}f_t^0\left(\sum_{s=1}^{t-1}e_s'\rho^{t-s}\right)\right\|^2 \\
= & \frac{1}{nT^2}trE\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}f_t^0e_s'\rho^{t-s}\rho^{t-q}e_qf_p^{0\prime} \\
= & \frac{1}{nT^2}trE\sum_{t=2}^{T}\sum_{p=2}^{T}f_t^0f_p^{0\prime}\left(\sum_{i=1}^{n}\sum_{s=1}^{t-1}\sum_{q=1}^{p-1}\rho_i^{2t-s-q}e_{is}e_{iq}\right)=O(1),
\end{aligned}$$

as required. ■

Proof of $\frac{\left\|e'Z_{-1}^0(\rho-I_n)+(\rho-I_n)Z_{-1}^{0\prime}e\right\|}{nT}=O_p\left(\frac{1}{\sqrt{n}T}\right)$

By definition

$$\frac{\left\|e'Z_{-1}^0(\rho-I_n)+(\rho-I_n)Z_{-1}^{0\prime}e\right\|}{nT}\leq 2\frac{\left\|e'Z_{-1}^0\Theta\right\|}{n\sqrt{n}T^2}.$$

And

$$\begin{aligned}
\frac{\left\|e'Z_{-1}^0\Theta\right\|}{n\sqrt{n}T^2} &= \frac{1}{n\sqrt{n}T^2}\left\|\sum_{t=2}^{T}e_tz_{t-1}^{0\prime}\Theta\right\| \\
&\leq \frac{1}{n\sqrt{n}T^2}\left\|\sum_{t=2}^{T}e_t\left(\sum_{s=1}^{t-1}f_s^{0\prime}\beta^{0\prime}\rho^{t-s}\right)\Theta\right\|+\frac{1}{n\sqrt{n}T^2}\left\|\sum_{t=2}^{T}e_t\left(\sum_{s=1}^{t-1}e_s'\rho^{t-s}\right)\Theta\right\|.
\end{aligned}$$

Then,

$$\begin{aligned}
& \frac{1}{n^3T^4}E\left\|\sum_{t=2}^{T}e_t\left(\sum_{s=1}^{t-1}f_s^{0\prime}\beta^{0\prime}\rho^{t-s}\right)\Theta\right\|^2 \\
= & \frac{1}{n^3T^4}trE\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}e_tf_s^{0\prime}\beta^{0\prime}\rho^{t-s}\Theta\right)\left(\sum_{p=2}^{T}\sum_{q=1}^{p-1}e_pf_q^{0\prime}\beta^{0\prime}\rho^{p-q}\Theta\right)' \\
= & \frac{1}{n^3T^4}trE\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}\beta^{0\prime}\rho^{t-s}\Theta^2\rho^{p-q}\beta^0f_q^0e_p'e_tf_s^{0\prime}\right) \\
= & \frac{1}{n^3T^4}tr\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}E\left(\sum_{i=1}^{n}e_{ip}e_{it}\right)E\left(\beta^{0\prime}\rho^{t-s}\Theta^2\rho^{p-q}\beta^0\right)E\left(f_q^0f_s^{0\prime}\right)\right) \\
= & O\left(\frac{1}{nT^2}\right).
\end{aligned}$$

Thus,

$$\frac{1}{n\sqrt{n}T^2}\left\|\sum_{t=2}^{T}e_t\left(\sum_{s=1}^{t-1}f_s^{0\prime}\beta^{0\prime}\rho^{t-s}\right)\Theta\right\|=O_p\left(\frac{1}{\sqrt{n}T}\right).$$

Next,

$$
\begin{aligned}
& \frac{1}{n^3T^4}E\left\|\sum_{t=2}^{T}e_t\left(\sum_{s=1}^{t-1}e_s'\rho^{t-s}\right)\Theta\right\|^2 \\
= & \frac{1}{n^3T^4}trE\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}e_te_s'\rho^{t-s}\Theta^2\rho^{p-q}e_qe_p'\right) \\
= & \frac{1}{n^3T^4}trE\left(\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{p=2}^{T}\sum_{q=1}^{p-1}e_s'\rho^{t-s}\Theta^2\rho^{p-q}e_qe_p'e_t\right) \\
= & O\left(\frac{1}{nT^2}\right).
\end{aligned}
$$

Thus,

$$
\frac{1}{n\sqrt{n}T^2}\left\|\sum_{t=2}^{T}e_t\left(\sum_{s=1}^{t-1}e_s'\rho^{t-s}\right)\Theta\right\| = O_p\left(\frac{1}{\sqrt{n}T}\right).
$$

And we have the required result that

$$
\frac{\left\|e'Z_{-1}^0\left(\rho-I_n\right)+\left(\rho-I_n\right)Z_{-1}^{0\prime}e\right\|}{nT} = O_p\left(\frac{1}{\sqrt{n}T}\right)
$$

3 Appendix D: Proofs for the results in Section 3

Proof of Lemma 5.
The proofs of Parts (a) and (b) are similar to those of Lemma 2(a) - (c), and we omit it.[6]

Part (c). Let $P_G = I_T - Q_G$. For a $T\times n$ matrix A, A_t' denotes the transpose of the t^{th} row of A. Then,

$$
\begin{aligned}
& \frac{1}{nT}tr\left(\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0(0)\right)Q_{\beta^0}\tilde{e}'\right) \\
= & \frac{1}{nT}tr\left(e'\left(Z_{-1}^0-Z_{-1}^0(0)\right)Q_{\beta^0}\right)-\frac{1}{nT}tr\left(e'P_G\left(Z_{-1}^0-Z_{-1}^0(0)\right)Q_{\beta^0}\right).
\end{aligned}
$$

In what follows we show that

$$
\frac{1}{\sqrt{n}T}tr\left(e'\left(Z_{-1}^0-Z_{-1}^0(0)\right)Q_{\beta^0}\right) = o_p(1), \tag{57}
$$

and

$$
\sqrt{n}\left[\frac{1}{nT}tr\left(e'P_G\left(Z_{-1}^0-Z_{-1}^0(0)\right)Q_{\beta^0}\right)+\left(\frac{\mu_\theta}{n^\eta}\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)h_{kT}(s,t)\right)\right] = o_p(1), \tag{58}
$$

Then, since $\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)h_{kT}(s,t)-\int_0^1\int_0^r(r-s)h_k(r,s)\,dsdr = O\left(\frac{1}{T}\right)$, we have the required result.

[6] Upon requestion, the details of the proofs could be obtained from the authors.

First, for (57),

$$
\begin{aligned}
& \frac{1}{\sqrt{n}T} tr\left(e'\left(Z_{-1}^0 - Z_{-1}^0(0)\right) Q_{\beta^0}\right) \\
= & \frac{1}{\sqrt{n}T} \sum_{t=2}^{T} \left(Z_{t-1}^0 - Z_{t-1}^0(0)\right)' Q_{\beta^0} e_t \sim -\frac{\mu_\theta}{n^{1/2+\eta}T} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \left(\frac{t-s-1}{T}\right) y_s' Q_{\beta^0} e_t \\
= & -\frac{\mu_\theta}{n^{1/2+\eta}T} \sum_{i=1}^{n} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \left(\frac{t-s-1}{T}\right) e_{is} e_{it} \\
& + \frac{\mu_\theta}{n^{1/2+\eta}T} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \left(\frac{t-s-1}{T}\right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_{is} \beta_i^{0\prime}\right) \left(\frac{1}{n} \sum_{i=1}^{n} \beta_i^{0\prime} \beta_i^0\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \beta_i^0 e_{it}\right) \\
= & I_a + II_a, \text{ say.}
\end{aligned}
$$

Since

$$
\begin{aligned}
E\left(I_a^2\right) &= \frac{\mu_\theta^2}{n^{2\eta}T^2} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \sum_{p=2}^{T} \sum_{q=1}^{t-1} \left(\frac{t-s-1}{T}\right) \left(\frac{p-q-1}{T}\right) E\left(e_{is} e_{it} e_{ip} e_{iq}\right) \\
&= \frac{\mu_\theta^2}{n^{2\eta}T^2} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \left(\frac{t-s-1}{T}\right)^2 = O\left(\frac{1}{n^{2\eta}}\right) = o(1),
\end{aligned}
$$

we have

$$
I_a = o_p(1).
$$

Also, similarly, it is possible to show that

$$
\begin{aligned}
E\left(II_a^2\right) &= E \frac{\mu_\theta^2}{n^{1+2\eta}T^2} \sum_{t=2}^{T} \sum_{s=1}^{t-1} \sum_{p=2}^{T} \sum_{q=1}^{t-1} \left(\frac{t-s-1}{T}\right) \left(\frac{p-q-1}{T}\right) \\
& \times \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_{is} \beta_i^{0\prime}\right) \left(\frac{1}{n} \sum_{i=1}^{n} \beta_i^{0\prime} \beta_i^0\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \beta_j^0 e_{jt}\right) \\
& \times \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{kq} \beta_k^{0\prime}\right) \left(\frac{1}{n} \sum_{i=1}^{n} \beta_i^{0\prime} \beta_i^0\right)^{-1} \left(\frac{1}{\sqrt{n}} \sum_{l=1}^{n} \beta_l^0 e_{lp}\right) \\
&= O\left(\frac{1}{n^{1+2\eta}}\right),
\end{aligned}
$$

so that

$$
II_a = o_p(1),
$$

which leads (57).

Next, for (58) we write

$$
\begin{aligned}
& \frac{1}{nT} tr\left(e' P_G\left(Z^0_{-1} - Z^0_{-1}(0)\right) Q_{\beta^0}\right) \\
= & \frac{1}{nT} tr\left(e' G_{kT}\left(G'_{kT} G_{kT}\right)^{-1} G'_{kT}\left(Z^0_{-1} - Z^0_{-1}(0)\right) Q_{\beta^0}\right) \\
= & \frac{1}{nT} tr\left(\sum_{t=1}^{T} e_t g'_{kt}\right)\left(\sum_{t=1}^{T} g_{kt} g'_{kt}\right)^{-1}\left(\sum_{t=1}^{T} g_{kt}\left(Z^0_{t-1} - Z^0_{t-1}(0)\right)' Q_{\beta^0}\right). \\
\sim & -\frac{\mu_\theta}{n^{1+\eta}T} tr\left(\sum_{t=1}^{T} e_t g'_{kt}\right)\left(\sum_{t=1}^{T} g_{kt} g'_{kt}\right)^{-1}\left(\sum_{t=2}^{T} g_{kt} \sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) e'_s Q_{\beta^0}\right) \\
& + \frac{1}{2}\frac{\mu_\theta^2}{n^{1+2\eta}T} tr\left(\sum_{t=1}^{T} e_t g'_{kt}\right)\left(\sum_{t=1}^{T} g_{kt} g'_{kt}\right)^{-1}\left(\sum_{t=3}^{T} g_{kt} \sum_{s=1}^{t-2}\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right) e'_s Q_{\beta^0}\right) \\
& - \frac{1}{6}\frac{\mu_\theta^3}{n^{1+3\eta}T} tr\left(\sum_{t=1}^{T} e_t g'_{kt}\right)\left(\sum_{t=1}^{T} g_{kt} g'_{kt}\right)^{-1}\left(\sum_{t=4}^{T} g_{kt} \sum_{s=1}^{t-3}\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)\left(\frac{t-s-3}{T}\right) e'_s Q_{\beta^0}\right) \\
= & I_b + II_b + III_b, \text{ say.}
\end{aligned}
$$

First, notice that

$$
\begin{aligned}
I_b = & -\frac{\mu_\theta}{n^{1+\eta}T^2} \sum_{p=1}^{T}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(p,t)\, e'_s Q_{\beta^0} e_p \\
= & -\frac{\mu_\theta}{n^{1+\eta}T^2} \sum_{p=1}^{T}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(p,t)\left(\sum_{i=1}^{n} e_{is} e_{ip}\right) \\
& + \frac{\mu_\theta}{n^{1+\eta}T^2} \sum_{p=1}^{T}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(p,t)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{is}\beta_i^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0\prime}\beta_i^0\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\beta_j^0 e_{jp}\right) \\
= & I_{b1} + I_{b2}, \text{ say.}
\end{aligned}
$$

A direct calculation shows that

$$
E\left(\sqrt{n} I_{b2}\right)^2 = O\left(\frac{1}{n^{1+2\eta}}\right),
$$

which yields

$$
\sqrt{n} I_{b2} = o_p(1).
$$

Also,

$$
\begin{aligned}
& E\left[\sqrt{n}\left\{I_{b1} + \left(\frac{\mu_\theta}{n^\eta}\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(s,t)\right)\right\}\right]^2 \\
= & \frac{\mu_\theta^2}{n^{2\eta}} E\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\begin{array}{c} \frac{1}{T^2}\sum_{p=1}^{T}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(p,t)\, e_{is} e_{ip} \\ -E\left\{\frac{1}{T^2}\sum_{p=1}^{T}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(p,t)\, e_{is} e_{ip}\right\} \end{array}\right\}\right]^2 = O\left(\frac{1}{n^{2\eta}}\right),
\end{aligned}
$$

so

$$
\sqrt{n}\left\{I_{b1} + \left(\frac{\mu_\theta}{n^\eta}\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right) h_{kT}(s,t)\right)\right\} = o_p(1),
$$

which yields

$$I_b = o_p\left(1\right).$$

Next, we split two cases. When $k=0$, $h_{0T}\left(p,t\right)=1$, and we have

$$\begin{aligned}
\sqrt{n}II_b &= \frac{\mu_\theta^2}{2}n^{1/2-2\eta}\frac{1}{nT^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)e_s'Q_{\beta^0}e_p \\
&= O_p\left(n^{1/2-2\eta}\right) = o_p\left(1\right)
\end{aligned}$$

because $\eta > \frac{1}{4}$ in the case of $k=0$. Similarly, we can show that

$$\sqrt{n}III_b = o_p\left(1\right).$$

For $k=1$, we write

$$\begin{aligned}
II_b &= \frac{\mu_\theta^2}{2}\frac{1}{n^{1+2\eta}T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)e_s'Q_{\beta^0}e_p \\
&= \frac{\mu_\theta^2}{2}\frac{1}{n^{1+2\eta}T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)\left(\sum_{i=1}^{n}e_{is}e_{ip}\right) \\
&\quad +\frac{\mu_\theta^2}{2}\frac{1}{n^{1+2\eta}T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right) \\
&\quad \times\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}e_{is}\beta_i^{0\prime}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^{0\prime}\beta_i^0\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\beta_j^0 e_{jp}\right) \\
&= II_{b1}+II_{b2},\ \text{say.}
\end{aligned}$$

For II_{b1}, we split the term again as

$$\begin{aligned}
II_{b1} &= II_{b1}-E\left(II_{b1}\right)+E\left(II_{b1}\right) \\
&= \frac{\mu_\theta^2}{2n^{2\eta}}\frac{1}{n}\sum_{i=1}^{n}\left\{\begin{array}{c} \frac{1}{T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)e_{is}e_{ip} \\ -E\left(\frac{1}{T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)e_{is}e_{ip}\right)\end{array}\right\} \\
&\quad +\frac{\mu_\theta^2}{2n^{2\eta}}\frac{1}{T^2}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(t,s\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right).
\end{aligned}$$

Then, for some constant M_1 and M_2, we have

$$\begin{aligned}
& E\left(\sqrt{n}II_{b1}\right)^2 \\
\leq\ & \frac{M_1}{n^{4\eta}}E\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\begin{array}{c} \frac{1}{T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)e_{is}e_{ip} \\ -E\left(\frac{1}{T^2}\sum_{p=1}^{T}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)e_{is}e_{ip}\right)\end{array}\right\}\right]^2 \\
& +\frac{M_2}{n^{4\eta}}\left(\frac{1}{T^2}\sum_{t=3}^{T}\sum_{s=1}^{t-2}h_{1T}\left(t,s\right)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right)\right) \\
=\ & O\left(\frac{1}{n^{4\eta}}\right)+o\left(\frac{1}{n^{4\eta}}\right)
\end{aligned}$$

because $\frac{1}{T^2}\sum_{t=3}^{T}\sum_{s=1}^{t-2} h_{1T}(t,s)\left(\frac{t-s-1}{T}\right)\left(\frac{t-s-2}{T}\right) \to \int_0^1\int_0^s (r-s)^2 h(r,s)\,dsdr = 0$. Therefore,

$$\sqrt{n}II_{b1} = o_p(1).$$

Finally, we can show that

$$E\left(\sqrt{n}III_b\right)^2 = n^{1-3\eta}O(1) = o(1),$$

which yields that

$$III_b = o_p(1).$$

Then, we have all the required results for Part (c). ■

Part (d). Here we show that

$$\frac{\sqrt{n}}{n^\eta}\left[\frac{tr\left(\tilde{Z}_{-1}(0)\,Q_{\beta^0}\tilde{Z}_{-1}'(0)\right)}{nT^2} - \left(\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T} - \frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=2}^{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right)h_{kT}(t,s)\right)\right] = o_p(1). \tag{59}$$

Then, since

$$\begin{aligned}
&\left(\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T} - \frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=2}^{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right)h_{kT}(t,s)\right)\\
=\;& \int_0^1 rdr - \int_0^1\int_0^1 \min(r,s)\,h(r,s)\,dsdr + O\left(\frac{1}{T}\right),
\end{aligned}$$

the proof of Part (d) is done.

Since $\tilde{Z}_{-1}(0) = \tilde{F}_{-1}^0\beta^{0\prime} + \tilde{E}_{-1}$, we have

$$\begin{aligned}
&\frac{1}{n^\eta\sqrt{n}T^2}tr\left(\tilde{Z}_{-1}^0(0)\,Q_{\beta^0}\tilde{Z}_{-1}^0(0)'\right)\\
=\;& \frac{1}{n^\eta\sqrt{n}T^2}tr\left(\tilde{E}_{-1}Q_{\beta^0}\tilde{E}_{-1}'\right)\\
=\;& \frac{1}{n^\eta\sqrt{n}T^2}tr\left(\tilde{E}_{-1}\tilde{E}_{-1}'\right) - \frac{1}{n^{1+\eta}\sqrt{n}T^2}tr\left(\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}\tilde{E}_{-1}'\tilde{E}_{-1}\beta^0\right)
\end{aligned}$$

Using similar arguments in the proof of Lemma 9(c), we can show that

$$\begin{aligned}
\frac{1}{n^{1+\eta}\sqrt{n}T^2}tr\left(\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}\tilde{E}_{-1}'\tilde{E}_{-1}\beta^0\right) &\leq \frac{1}{n^{1/2+\eta}}\left\|\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\right\|\frac{\left\|\beta^{0\prime}\tilde{E}_{-1}'\right\|}{\sqrt{n}T}\frac{\left\|\tilde{E}_{-1}\beta^0\right\|}{\sqrt{n}T}\\
&= O_p\left(\frac{1}{n^{1/2+\eta}}\right).
\end{aligned}$$

Therefore, to have the required result (59), it is enough to show

$$\frac{\sqrt{n}}{n^\eta}\left[\frac{tr\left(\tilde{E}_{-1}\tilde{E}_{-1}'\right)}{nT^2} - \left(\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T} - \frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=2}^{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right)h_{kT}(t,s)\right)\right] = o_p(1),$$

which follows by a direct calculation that leads to

$$
\begin{aligned}
& E\left[\frac{\sqrt{n}}{n^{\eta}}\left\{\frac{tr\left(\tilde{E}_{-1}\tilde{E}_{-1}'\right)}{nT^2}-\left(\frac{1}{T}\sum_{t=2}^{T}\frac{t-1}{T}-\frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=2}^{T}\min\left(\frac{t-1}{T},\frac{s-1}{T}\right)h_{kT}\left(t,s\right)\right)\right\}\right]^2 \\
= \; & O\left(\frac{1}{n^{2\eta}}\right)=o\left(1\right),
\end{aligned}
$$

and we have all the required result.

Part (e). Notice that

$$\left\|Z_{-1}^0-Z_{-1}^0\left(0\right)\right\|=O_p\left(Tn^{1/2-\eta}\right)$$

because

$$
\begin{aligned}
\frac{1}{n^{1-2\eta}T^2}\left\|Z_{-1}^0-Z_{-1}^0\left(0\right)\right\|^2 &= \frac{1}{n^{1-2\eta}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(z_{it-1}^0-z_{it-1}^0\left(0\right)\right)^2 \\
\sim \; \frac{\mu_\theta^2}{nT^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)y_{is}\right)^2 &= O_p\left(1\right)
\end{aligned}
$$

by Lemma 10(a) and (b). Then, the required result follows because for some constant $M>0$,

$$\frac{1}{n^{1/2+\eta}T^2}tr\left(\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0\left(0\right)\right)Q_{\beta^0}\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0\left(0\right)\right)'\right)\leq n^{1/2-3\eta}M\frac{\left\|Z_{-1}^0-Z_{-1}^0\left(0\right)\right\|^2}{n^{1-2\eta}T^2}=O_p\left(n^{1/2-3\eta}\right)$$

and since $1/6<\eta$ in both model $k=0$ and $k=1$, $n^{1/2-3\eta}=o\left(1\right)$, as required. ■

Part (f). Define

$$E_{it}\left(\mu_\theta\right)=\sum_{s=1}^{t}\left(1-\frac{\mu_\theta}{n^\eta T}\right)^{t-s}e_{is}.$$

Let

$$
\begin{aligned}
E_t\left(\mu_\theta\right)' &= \left(E_{1t}\left(\mu_\theta\right),...,E_{nt}\left(\mu_\theta\right)\right), \\
E_{-1}\left(\mu_\theta\right) &= \left(0',E_1\left(\mu_\theta\right)',...,E_{T-1}\left(\mu_\theta\right)'\right)'.
\end{aligned}
$$

By definition,

$$E_{-1}\left(0\right)=E_{-1}.$$

Using this notation, write

$$
\begin{aligned}
& \frac{1}{n^{1/2+\eta}T^2}tr\left(\left(\tilde{Z}_{-1}^0-\tilde{Z}_{-1}^0\left(0\right)\right)Q_{\beta^0}\tilde{Z}_{-1}'\left(0\right)\right) \\
= \; & \frac{1}{n^{1/2+\eta}T^2}tr\left(Q_G\left(E_{-1}\left(\mu_\theta\right)-E_{-1}\right)Q_{\beta^0}E_{-1}'Q_G\right) \\
= \; & \frac{1}{n^{1/2+\eta}T^2}tr\left(E_{-1}'Q_G\left(E_{-1}\left(\mu_\theta\right)-E_{-1}\right)\right)-\frac{1}{n^{1/2+\eta}T^2}tr\left(Q_G\left(E_{-1}\left(\mu_\theta\right)-E_{-1}\right)P_{\beta^0}E_{-1}'Q_G\right) \\
= \; & I_f-II_f, \text{ say.}
\end{aligned}
$$

Then,

$$
\begin{aligned}
II_f &= \frac{1}{n^{3/2+\eta}T^2} tr\left(\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}E_{-1}^{\prime}Q_G\left(E_{-1}\left(\mu_\theta\right)-E_{-1}\right)\beta^0\right) \\
&= \frac{1}{n^{3/2+\eta}T^2} tr\left(\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}E_{-1}^{\prime}\left(E_{-1}\left(\mu_\theta\right)-E_{-1}\right)\beta^0\right) \\
&\quad -\frac{1}{n^{3/2+\eta}T^2} tr\left(\left(\frac{\beta^{0\prime}\beta^0}{n}\right)^{-1}\beta^{0\prime}E_{-1}^{\prime}G_{kT}\left(G_{kT}^{\prime}G_{kT}\right)^{-1}G_{kT}^{\prime}\left(E_{-1}\left(\mu_\theta\right)-E_{-1}\right)\beta^0\right) \\
&= -\frac{\mu_\theta}{n^{3/2+2\eta}T^2} tr\left(\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_i^0\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)E_{it-1}e_{js}\beta_j^{0\prime}\right) \\
&\quad +\frac{\mu_\theta}{n^{3/2+2\eta}T^3} tr\left(\left(\frac{1}{n}\sum_{i=1}^{n}\beta_i^0\beta_i^{0\prime}\right)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=2}^{T}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\beta_i^0E_{ip-1}h_{k,T}\left(p,t\right)\left(\frac{t-s-1}{T}\right)e_{js}\beta_j^{0\prime}\right) \\
&= II_{fa}-II_{fb},\ \text{say.}
\end{aligned}
$$

A direct calculation shows that

$$
II_{fa}=O_p\left(\frac{1}{n^{1+2\eta}}\right)=o_p\left(1\right)\ \text{ and } II_{fb}=O_p\left(\frac{1}{n^{1+2\eta}}\right)=o_p\left(1\right).
$$

Also, for I_f,

$$
\begin{aligned}
I_f &= -\frac{\mu_\theta}{n^{1/2+2\eta}T^2}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)e_{is}\right)E_{it-1} \\
&\quad +\frac{\mu_\theta}{n^{1/2+2\eta}T^3}\sum_{i=1}^{n}\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)e_{is}\right)\sum_{p=2}^{T}E_{ip-1}h_{k,T}\left(t,p\right)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
& E\left(\frac{1}{T^2}\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)e_{is}\right)E_{it-1}-\frac{1}{T^3}\sum_{t=2}^{T}\left(\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)e_{is}\right)\sum_{p=2}^{T}E_{ip-1}h_{k,T}\left(t,p\right)\right) \\
&= \frac{1}{T^2}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\left(\frac{t-s-1}{T}\right)-\frac{1}{T^3}\sum_{t=2}^{T}\sum_{p=2}^{T}h_{k,T}\left(t,p\right)\sum_{s=1}^{\min(t,p)-1}\left(\frac{t-s-1}{T}\right) \\
&= \int_0^1\int_0^r\left(r-s\right)dsdr-\int_0^1\int_0^1 h_k\left(r,p\right)\int_0^{\min(r,p)}\left(r-s\right)dsdpdr+O\left(\frac{1}{T}\right).
\end{aligned}
$$

Notice that a direct calculation shows that

$$
\int_0^1\int_0^r\left(r-s\right)dsdr-\int_0^1\int_0^1 h_k\left(r,p\right)\int_0^{\min(r,p)}\left(r-s\right)dsdpdr=\frac{1}{8}\ \text{for } k=0\ (\text{so that } h_k\left(r,p\right)=1)
$$

and

$$
\int_0^1\int_0^r\left(r-s\right)dsdr-\int_0^1\int_0^1 h_k\left(r,p\right)\int_0^{\min(r,p)}\left(r-s\right)dsdpdr=0\ \text{for } k=1.
$$

Then, for model $k=0$, it is possible to show that

$$I_f = -\mu_\theta n^{1/2-2\eta} \left[\begin{array}{c} \frac{1}{nT^2} \sum_{i=1}^{n} \sum_{t=2}^{T} \left(\sum_{s=1}^{t-1} \left(\frac{t-s}{T} \right) e_{is} \right) E_{it-1} \\ -\frac{1}{nT^3} \sum_{i=1}^{n} \sum_{t=2}^{T} \left(\sum_{s=1}^{t-1} \left(\frac{t-s}{T} \right) e_{is} \right) \sum_{p=2}^{T} E_{ip-1} h_{k,T}(t,p) \end{array} \right] = O_p \left(n^{1/2-2\eta} \right) = o_p(1)$$

because $\eta > \frac{1}{4}$, and for model $k=1$, it is possible to show that

$$E\left(I_f^2\right) = O\left(\frac{1}{n^{4\eta}}\right) = o(1),$$

so

$$I_f = o_p(1),$$

as required. ■

References

[1] Moon, H.R. and B. Perron (2003): Testing for A Unit Root in Panels with Dynamic Factors, *CLEO Discussion Papers*, University of Southern California.