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USC Center for Law, Economics & Organization Research Paper No. C03-22

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Sponsored by the John M. Olin Foundation

University of Southern California Law School
Los Angeles, CA 90089-0071

Dynamic Principal-Agent Problems with Perfect Information *

Abel Cadenillas[†]

Jakša Cvitanić [‡]

Fernando Zapatero[§]

This draft: January 19, 2005

Abstract

We consider a continuous-time setting, in which the agent can control both the drift and the volatility of the underlying process. The principal can observe the agent's action and can offer payment at a continuous rate, as well as a bulk payment at the end of the fixed time horizon. In examples, we show that if the principal and the agent have the same CRRA utility, or they both have (possibly different) CARA utilities, the optimal contract is (ex-post) linear; if they have different CRRA utilities, the optimal contract is nonlinear, and can be of the call option type. We use martingale/duality methods, which, in the general case, lead to the optimal contract as a fixed point of a functional that connects the agent's and the principal's utility maximization problems.

Keywords: Principal-Agent Problem, Martingale Methods

JEL classification: C61, J33

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*The research of A. Cadenillas was supported by the Social Sciences and Humanities Research Council of Canada grant 410 - 2003 - 1401. The research of J. Cvitanić was supported in part by the National Science Foundation, under Grant NSF-DMS-00-99549. We are very grateful to two anonymous referees whose suggestions led to the great improvement in the exposition of the results, as well as to the introduction of the dividend rate D and the compensation rate q.

[†]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada. Ph: (780)492-0572. Fax: (780)492-6826. E-mail: acadenil@math.ualberta.ca.

[‡]Departments of Mathematics and Economics, USC, 3620 S Vermont Ave, MC 2532, Los Angeles, CA 90089. Ph: (213) 740-3794. Fax: (213) 740-2424. E-mail: cvitanic@math.usc.edu.

[§]FBE, Marshall School of Business, USC, Los Angeles, CA 90089-1427. Ph: (213) 740-6538. Fax: (213) 740-6650. E-mail: fzapatero@marshall.usc.edu.

1 Introduction

Principal-agent models are suitable for a large number of problems in economics in general, and in financial economics in particular. The vast majority of these models, however, are static. In recent times, a few papers have tried to model principal-agent problems in a dynamic setting, both in multiperiod discrete-time and continuous-time settings. In all these models there is a stochastic process that is part of the utility of the principal and whose dynamics the agent can affect. This stochastic process can represent, for example, the value of a project, the value of a company, the price of a share of stock, the value of a portfolio of securities, among other possibilities. In discrete-time, this stochastic process is always modelled as draw from a stationary conditional (on the action of the agent) distribution; in continuous-time, as a diffusion process adapted to the filtration generated by a (possible multidimensional) Brownian motion process and, depending on the problem, the agent can affect the drift and/or volatility of the process. Also, the agent might incur a cost as a result of the effort. Many of these models assume that the principal and the agent have full information. On the other hand, in most of the explicitly solved examples in the case of partial (hidden) information, the solution is static, i.e., the optimal control is constant. Finally, when analytic solutions are feasible, they typically imply a linear contract. We only consider the case of full information in this paper. Even for this case there are many applications, including problems in "delegated portfolio management" and optimal executive compensation.

In this paper we develop and analyze a continuous-time framework with full information. We mention here some related papers in discrete-time: Spear and Srivastava (1987) characterize the general solution of a dynamic problem with hidden action, using dynamic programming principles; Phelan and Towsend (1991) present a numerical algorithm to solve a general set of dynamic problems with hidden action; DeMarzo and Fishman (2003) apply the dynamic programming results of the former to a large number of problems affecting the dynamics of a firm. With respect to the continuous-time literature, to which this paper belongs, we start with the papers on the hidden information case. The pioneering paper in this setup is Holmström and Milgrom (1987). In that paper the agent controls only the drift. They show that if both the principal and the agent have exponential utilities, then the optimal contract is linear. Subsequently, Schättler and Sung (1993) generalized results of Holmström and Milgrom (1987) using the dynamic programming and martingales approaches of Stochastic Control Theory, and Sung (1995) showed that the linearity of the optimal contract still holds when the agent can control the volatility as well. Müller (1998, 2000) finds the full information (the "first-best") solution in the exponential case, and shows how it can be approximated by control revisions taking place at discrete times.

A connection with discrete-time models is developed in Hellwig and Schmidt (2003). More complex models are considered in a recent paper Bolton and Harris (2001). Also recently, Detemple, Govindaraj and Loewenstein (2001) provide a much more general framework in which the agent controls the drift only. Very recently Williams (2003) also considers general utilities in Markovian models driven by Brownian Motion. That paper uses a different approach, the stochastic maximum principle, in order to characterize the optimal contract in the principal-agent problems with hidden information, in the case of the penalty on the agent's effort being separate (outside) of his utility, and without volatility control.

A recent dynamic principal-agent model with full information and applications to the problem of money-management compensation is Ou-Yang (2003). In this paper, the agent allocates money across different securities, which is equivalent to controlling the volatility of the wealth process of the principal. Ou-Yang (2003) deals with two types of problems: one with a more elaborate model and the cost function, but with exponential utilities; another with a simpler model and cost function, but general utilities.

In this paper, we use a technical approach different from the standard Hamilton-Jacobi-Bellman dynamic programming approach used in most of the above mentioned papers (Ou-Yang 2003 also uses an approach similar to ours in a particular case considered in his paper). This approach is more general, and most of the models with full information listed above are particular cases of the setting considered in this paper. However, we do not consider here the hidden information case. We extend existing results with full information in several directions. Most importantly, our approach allows us to have arbitrary utility functions for the principal and the agent. Also, our approach can, in principle, be applied to general semimartingale models, and not only to Markovian, diffusion models. Moreover, the agent controls both the volatility and the drift in our model, and we allow an interplay between the volatility and the drift of the underlying process (Ou-Yang 2003 is the only paper in the previous list that allows this interplay, but it allows no independent control of drift). In addition, we allow that the agent may be paid continuously, and that the principal can consume a dividend from the underlying process. The generality of our model makes it suitable for many applications as particular cases, including, as mentioned above, optimal executive compensation and money-management compensation.

An important result of our paper is that when the principal and the agent have different utility functions outside the class of CARA utilities, the optimal contract is nonlinear and/or path dependent. Moreover, a wide range of contracts can be optimal, and we construct an example in which a call option contract is optimal.

The above results are presented in a model driven by only one Brownian Motion, and with a cost function depending only on the drift control. In the final part of the paper we set up a general semimartingale model and a general cost function, and describe an approach that could be applied in such a context. The approach is based on searching for a payoff which is a “fixed point” of a functional that connects the agent’s and the principal’s utility maximization problems. We re-derive the solution to the main example solved in Ou-Yang (2003) using this approach. We also address the original Holmström-Milgrom problem with exponential utilities and controlling the drift only. We show that in our case, in which the principal observes the underlying Brownian Motion and not only the underlying controlled process, the principal’s optimal utility is larger than his optimal utility when he can only offer contracts based only on the underlying controlled process.

The paper is organized as follows: In Section 2 we set up the model. In Section 3 we first solve the so-called “first-best” problem, in which the principal controls all the actions, and finds the optimal ones. We accomplish this by using the techniques from the literature on portfolio optimization in complete markets, as developed initially by Pliska (1986), Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987), and presented in great generality in Karatzas and Shreve (1998). We then show that those first-best actions can be implemented by a contract of a simple form. We also discuss conditions under which the optimal contract depends only on the final value of the controlled process, in the case in which the drift is not controlled separately (with delegated portfolio management as the main application). We provide some explicit examples in Section 4. In Section 5 we present the general model with a general cost function, and solve some examples based on the techniques of portfolio optimization in markets with frictions, using ideas from Cvitanić (1997) and Cuoco and Cvitanić (1998). We conclude in Section 6, mentioning possible further research topics. The proofs of the main results are collected in the Appendix.

2 The Model

We introduce a continuous-time model driven by a single Brownian Motion. Later on we extend our approach to much more general models. Let W be a Brownian Motion process on a probability space $(\Omega, \mathcal{F}, P)$ and denote by $\mathbf{F} := \{\mathcal{F}_t\}_{t \le T}$ its augmented filtration on the interval $[0, T]$. Let us call the process controlled by the agent “stock price”, motivated by the example of a company compensating its executives. Another typical example would be an example of a portfolio manager managing a portfolio S. The dynamics of the process $S = S^{a,\sigma,D}$ are given by

$$dS_t \; = \; \delta a_t dt - D_t dt + \alpha \sigma_t dt + \sigma_t dW_t, \tag{2.1}$$

where $\delta \in [0, \infty)$, $\alpha \in (0, \infty)$ are constants, and a, σ and D are $\mathbf{F}$-adapted stochastic processes chosen by the agent. Here D_t represents the “dividend” rate or the “consumption” rate of the principal. The control a is the level of effort the agent applies to his projects. The higher a, the higher the expected value of the stock. We will assume later that the effort produces disutility for the agent. On the other hand, the choice of σ is equivalent to the choice of the volatility of the stock, although it also has an impact on the expected value. We interpret the choice of σ as a choice of projects. We assume that the agent can choose different projects or strategies that are characterized by a level of risk and expected return. Since $\alpha > 0$, the higher the risk of a project, the higher its expected return. We note that in the case of delegated portfolio management the value of δ and D would be zero. We study that case separately in a later section.

The agent receives final payment P_T from the principal, as well as a continuous payment at a "compensation" rate q_t. Here, P_T is an $\mathcal{F}_T-$measurable random variable, while q is an adapted process. The agent's problem is to maximize, over a, σ and D,

$$E\left[U_1\left(P_T - \int_0^T G(a_s)ds\right) + \int_0^T V_1(q_s)ds\right] \quad . \tag{2.2}$$

Here, U_1 and V_1 are the utility functions of the agent, which we assume to be differentiable, strictly increasing and strictly concave. The function G measures the disutility from the effort, and we assume that $G(0) = 0$ and G is a strictly convex and differentiable function, strictly increasing in $|u|$.

The principal's problem is to maximize, over P_T and q_t,

$$E\left[U_2(\beta S_T - P_T) + \int_0^T V_2(\kappa D_s - q_s)ds\right] \quad . \tag{2.3}$$

In other words, the principal's utility measures the trade-off between the value of the stock at time T and the payoff P_T to the agent, as well as between the dividend rate D_t and the compensation rate q_t, where the relative importance of S_T and D_t is measured by constants $\beta > 0$, $\kappa > 0$. Here, $(1 - \kappa)$ can also account for the tax rate on dividends. The principal has full information, and in particular, can observe the agent's actions. The time horizon T is fixed.

We assume that U_2 and V_2 are strictly increasing and strictly concave functions.

The principal has to guarantee that the (optimal) utility (2.2) of the agent is at least as large as a reservation utility R. That is,

$$\max_{a,\sigma,D} E\left[U_1\left(P_T - \int_0^T G(a_s)ds\right) + \int_0^T V_1(q_s)ds\right] \geq R.$$

This can be interpreted as the utility that the agent would achieve in the best alternative offer he has. This restriction amounts to an *individual rationality constraint* or *participation constraint*, standard in the principal-agent literature. We will call it "IR constraint".

3 The First-Best Solution

We first solve the so-called "first-best" problem, in which the principal can force the agent to apply controls a, σ and D which are optimal for the principal's problem, as long as the IR constraint is satisfied. In order to describe the solution, denote by I_i^U the inverse function of the marginal utility function U_i', by I_i^V the inverse function of the marginal utility function V_i', and by J the inverse function of G' (they exist because U_i, V_i are strictly concave, and G is strictly convex). That is, for $f = U, V$,

$$I_i^f(z) := (f_i')^{-1}(z) \qquad \text{and} \qquad J(x) := (G')^{-1}(x). \tag{3.1}$$

Also introduce the auxiliary exponential martingale

$$Z_t = \exp\left\{-\alpha W_t - \frac{\alpha^2}{2}t\right\}$$

satisfying

$$dZ_t = -\alpha Z_t dW_t \quad .$$

We will see below that the IR constraint becomes, for some constant $\hat{z}$,

$$R \le E\left[U_1(I_1^U(\hat{z}Z_T)) + \int_0^T V_1\left(I_1^V(\frac{\beta}{\kappa}\hat{z}Z_s)\right)ds\right] \quad . \tag{3.2}$$

We need the following assumption to solve the principal's first-best problem

Assumption 3.1 *There exists a unique number $\hat{z}$ such that* (3.2) *is satisfied as equality.*

Moreover, we will need to use the so-called Martingale Representation Theorem to identify the optimal control σ. In this regard, note that, using Itô's rule, we have

$$M_t := S_t Z_t - \delta\int_0^t Z_s a_s ds + \int_0^t Z_s D_s ds = S_0 + \int_0^t (\sigma_s - \alpha S_s) Z_s dW_s. \tag{3.3}$$

That is, the left-hand side process M_t is a local martingale. We will impose a bit stronger (technical) requirement:

Definition 3.1 *The set of admissible actions is the set of triplets (a, σ, D) such that process M is a martingale.*

We also impose the following assumption, whose meaning will become more clear in the proof of Theorem 3.1 in the Appendix:

Assumption 3.2 *There exists a number $y = \hat{y}$ has so that the following principal's feasibility constraint is satisfied:*

$$\begin{aligned} &\beta(S_0 + T\delta J(\delta\beta)) \\ &= E\left[Z_T\left\{I_1^U(\hat{z}Z_T) + I_2^U(\hat{y}Z_T) + TG(J(\delta\beta))\right\} + \int_0^T \left\{Z_s\frac{\beta}{\kappa}\left[I_1^V\left(\frac{\beta}{\kappa}\hat{z}Z_s\right) + I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_s\right)\right]\right\} ds\right]. \end{aligned} \tag{3.4}$$

The main result in this subsection is

Theorem 3.1 *Suppose that Assumptions 3.1 and 3.2 hold. Then, the first-best solution consists of the payoff*

$$\hat{P}_T = \beta S_T - I_2^U(\hat{y}Z_T) \quad , \tag{3.5}$$

and the compensation rate

$$\hat{q}_t = \kappa D_t - I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_t\right) \quad . \tag{3.6}$$

Optimal D is given by

$$\kappa\hat{D}_t = I_1^V\left(\frac{\beta}{\kappa}\hat{z}Z_t\right) + I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_t\right) \quad , \tag{3.7}$$

optimal a is given by

$$\hat{a} \equiv J(\delta\beta) \quad , \tag{3.8}$$

and optimal σ is chosen so that at the final time we have

$$\beta\hat{S}_T = I_1^U(\hat{z}Z_T) + I_2^U(\hat{y}Z_T) + TG(J(\delta\beta)) \quad . \tag{3.9}$$

Proof: See the Appendix. ◇

Remark 3.1 (i) In all of our examples below, Assumptions 3.1 and 3.2 are satisfied.
(ii) In the case of hidden information, more typical in the principal-agent literature, it is assumed that the principal does not observe a and hence W is not observable either, and then our optimal contract $(\hat{P}, \hat{q})$ may not be feasible, since the process Z depends on W. However, notice that if S is continuously observable, then the control σ is observable, as a quadratic variation of process S. Thus, if there is no effort a, our solution is feasible.

3.1 Contracts that implement the first-best solution

We say that the actions a, σ, D are *implementable* by a contract (P_T, q) if, when offered that contract, the agent chooses optimally the actions a, σ, D. We now claim that the contract from Theorem 3.1 implements the first-best controls $\hat{a}, \hat{\sigma}, \hat{D}$.

Proposition 3.1 *We suppose that Assumptions 3.1, 3.2 hold. If the principal offers the agent the contract $(\hat{P}_T, \hat{q}_t)$ of* (3.5) *and* (3.6)*, then the agent will choose the first-best controls $\hat{a}, \hat{\sigma}, \hat{D}$ of Theorem 3.1.*

Proof: See the Appendix. $\diamond$

We discuss next when the optimal contract depends only on the final value S_T.

3.2 Payoff as a Function of S_T. The Case $\delta = D = q = 0$

We assume here that there is no effort a, and no dividend rate D and compensation rate q, and we denote $I_i^U = I_i$. (The main application is the compensation of portfolio managers.)

Remark 3.2 From (3.5) and (3.9), we need to have

$$\beta S_T - I_1(\hat{z}Z_T) \;=\; I_2(\hat{y}Z_T) + TG(J(\delta\beta)) \tag{3.10}$$

Suppose we can solve this equation for a unique positive solution Z_T in terms of S_T, as some function

$$Z_T \;=\; h(S_T) \;\; .$$

That will give us the form of the optimal contract in terms of S_T, using (3.5):

$$\hat{P}_T \;=\; f(S_T) \;:=\; \beta S_T - I_2(\hat{y}h(S_T)) \;=\; I_1(\hat{z}h(S_T)) + TG(J(\delta\beta)) \;\; . \tag{3.11}$$

Even though, in this case, the optimal payoff $\hat{P}_T$ turns out, ex post, to be of the form $P_T = f(S_T)$ for some function f, it is not true in general, that the original optimal contract is offered to the agent in that form, ex ante. More precisely, the principal does not offer the agent to pay her as $P_T = f(S_T)$, but as $\hat{P}_T = \beta S_T - I_2(\hat{y}Z_T)$. We show in the next section in a counterexample that the utility for the principal when it offers the contract $f(S_T)$ can be smaller than its optimal utility $E[U_2(I_2(\hat{y}Z_T))]$ resulting from the optimal contract of the above theorem, even when f is linear, if $\delta > 0$. On the other hand, if $\delta = 0$, we show that if $f(S_T)$ is a linear function, then the contract can be offered as $\hat{P}_T = f(S_T)$.

We have the following result about the optimal contract being a function of S_T:

Proposition 3.2 *Assume that $\delta = D = q = 0$, and the agent is allowed to use only controls σ for which the local martingale process ZS is a martingale. Assume also that there exists a function $d = d(s)$ which satisfies the ordinary differential equation*

$$U_1'(d(s))d'(s) \;=\; \frac{z_d}{\hat{y}}U_2'(\beta s - d(s)), \tag{3.12}$$

for a given constant z_d and $\hat{y}$ as before. Also assume that the maximum in

$$\bar{U}_1^d(z) \;=\; \max_s\{U_1(d(s)) - zs\} \tag{3.13}$$

is attained at a unique value $s = s(z)$ for which the first derivative of the function on the right-hand side is zero:

$$U_1'(d(s(z)))d'(s(z)) \; = \; z \quad . \tag{3.14}$$

Assume that z_d and the boundary condition for the solution $d(s)$ of the above ODE can be chosen so that

$$E[Z_T s(z_d Z_T)] \; = \; S_0$$

and

$$E[U_1(d(s(z_d Z_T)))] \; = \; R \quad . \tag{3.15}$$

Then, it is optimal for the principal to offer the contract

$$\hat{P}_T \; = \; d(S_T) \quad .$$

Proof: See the Appendix. $\diamond$

Remark 3.3 It can be checked that if both agents have exponential utilities, or if they both have the same power utility, then the solution to the ODE (3.12) is a linear function, so that the optimal contract is then linear. Moreover, there are many cases in which the solution exists and is not linear. We postpone the details until we present examples in the next section.

Next, we want to show, using a different argument, a related result: if the function f of (3.11) is linear, then the optimal contract can be offered as $f(S_T)$. First, we have the following corollary to Theorem 3.1.

Corollary 3.1 *Let assumptions of Theorem 3.1 hold and $\hat{y}$ and $\hat{z}$ be as in that theorem. If $\delta = D = q = 0$, any contract for which at time T we have*

$$P_T \; = \; \beta S_T - I_2(\hat{y} Z_T) \tag{3.16}$$

is optimal. Reversely, if a contract is optimal then it has to satisfy the above equality at time T.

Proof: See the Appendix. $\diamond$

Theorem 3.2 *Assume that $\delta = D = q = 0$, that the agent is allowed to use only controls σ for which the local martingale process ZS is a martingale, and that there exists a number $\hat{y}$ such that the principal's feasibility constraint* (3.4) *is satisfied, with $\hat{z}$ determined from the IR constraint* (3.2) *satisfied as equality. Assume that the function $f(s) = \beta s - I_2(\hat{y}h(s))$ of* (3.11) *is a non-constant linear function, and that there is a unique solution z^* to the equation*

$$S_0 \; = \; E[Z_T f^{-1}(I_1(z^* Z_T))]. \tag{3.17}$$

Then the contract can be offered in the linear form $P_T = f(S_T)$.

Proof: See the Appendix. $\diamond$

4 Examples

We only look at examples with $D = q = 0$, but they are easily extended to the general case.

Example 4.1 (*Power and Exponential Utilities.*) Consider first the case of power utilities in which, for $\gamma_1 < 1$, $\gamma_2 < 1$,

$$U_1(x) = \frac{1}{\gamma_1} x^{\gamma_1} \qquad \text{and} \qquad U_2(x) = \frac{1}{\gamma_2} x^{\gamma_2} \quad .$$

Then,

$$I_1(z) = z^{\frac{1}{\gamma_1 - 1}} \qquad \text{and} \qquad I_2(z) = z^{\frac{1}{\gamma_2 - 1}}.$$

Thus, (3.10) becomes

$$\beta S_T - (zZ_T)^{\frac{1}{\gamma_1 - 1}} \; = \; (yZ_T)^{\frac{1}{\gamma_2 - 1}} + TG(J(\delta\beta)) \quad .$$

In particular, if the utilities are the same, $\gamma_2 = \gamma_1$, we get

$$Z_T^{\frac{1}{\gamma_1 - 1}} \; = \; \frac{\beta S_T - TG(J(\delta\beta))}{z^{\frac{1}{\gamma_1 - 1}} + y^{\frac{1}{\gamma_1 - 1}}}$$

and

$$P_T \; = \; \beta S_T - I_2(yZ_T) \; = \; z^{\frac{1}{\gamma_1 - 1}} \frac{\beta S_T - TG(J(\delta\beta))}{z^{\frac{1}{\gamma_1 - 1}} + y^{\frac{1}{\gamma_1 - 1}}} + TG(J(\delta\beta)) \quad .$$

That is, if the principal and agent have the same utility, and they both behave optimally, the payoff turns out to be linear at time T. The payoff can be offered in this form if $\delta = 0$.* We note that if the principal and the agent have different power utilities, the solution will be a non-linear contract in general.

Consider now the case of exponential utilities in which

$$U_1(x) = -\frac{1}{\gamma_1} e^{-\gamma_1 x} \qquad \text{and} \qquad U_2(x) = -\frac{1}{\gamma_2} e^{-\gamma_2 x} \quad .$$

Then,

$$I_1(z) = -\frac{1}{\gamma_1} \log(z) \qquad \text{and} \qquad I_2(z) = -\frac{1}{\gamma_2} \log(z).$$

Thus, (3.10) becomes

$$\beta S_T + \frac{1}{\gamma_1} \log(zZ_T) \; = \; -\frac{1}{\gamma_2} \log(yZ_T) + TG(J(\delta\beta)) \quad .$$

*Ross (1973) shows this result in a static setting. He calls it “the principle of similarity.”

Solving for $\log(Z_T)$ in terms of S_T, we get

$$\log(Z_T) \;=\; \frac{TG(J(\delta\beta)) - \beta S_T - \frac{1}{\gamma_1}\log(z) - \frac{1}{\gamma_2}\log(y)}{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}}.$$

Then, using $P_T = I_1(zZ_T) + \int_0^T G(\hat{a}_s)ds$, we get

$$P_T \;=\; -\frac{1}{\gamma_1}\log(z) - \frac{1}{\gamma_1}\left\{\frac{TG(J(\delta\beta)) - \beta S_T - \frac{1}{\gamma_2}\log(y) - \frac{1}{\gamma_1}\log(z)}{\frac{1}{\gamma_2}+\frac{1}{\gamma_1}}\right\} + TG(J(\delta\beta)) \;\;.$$

That is, if they have exponential utilities, when they both behave optimally the payoff turns out to be linear at time T, even when they have different risk aversions. This is consistent with the results of Sung (1995) (although they are derived in the case of hidden information), who also considers exponential utilities, using a different approach. In his setting $\alpha = 0$. Here, this means that $Z_t \equiv 1$, so that the optimal contract is $\beta S_T - I_2(\hat{y})$.

Example 4.2 (*A Counterexample for Linear Payoffs.*) We show here with an example the case discussed in Remark 3.2, where we argued that when $\delta > 0$ it may not be optimal to assign the contract $f(S_T)$ from (3.11), even when it is linear. Consider exponential utilities as in the previous example. Then the function f is linear,

$$f(S_T) \;=\; c + bS_T,$$

where c and b can be determined from that example:

$$c = \frac{\gamma_2^{-1}}{\gamma_2^{-1}+\gamma_1^{-1}}\left[TG(J(\delta\beta)) + \gamma_1^{-1}\log(\hat{y}/\hat{z})\right] \qquad \text{and} \qquad b = \beta\frac{\gamma_1^{-1}}{\gamma_2^{-1}+\gamma_1^{-1}}. \tag{4.1}$$

Let us suppose now that the contract is offered as $f(S_T)$. We show in the Appendix that the principal's utility is then of the form

$$E\left[U_2(\beta S_T - P_T)\right] \;=\; -\frac{\hat{y}}{\gamma_2}\exp\left\{-\gamma_1\left[TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}/\tilde{z})\right]\right\} \tag{4.2}$$

and that this is strictly smaller than $-\frac{\hat{y}}{\gamma_2}$, which is the utility with the optimal contract $\hat{P}_T = \beta S_T - I_2(\hat{y}Z_T)$.

Example 4.3 (*Nonlinear Payoff as the ODE Solution.*) We show here by example that the ODE (3.12) can have a nonlinear solution, and hence that a payoff which is a nonlinear function of S_T can be optimal.

We assume that

$$U_1(x) = \log(x) \qquad \text{and} \qquad U_2(x) = -e^{-x} \;\;.$$

Then the ODE (3.12) becomes

$$e^{-d(s)}\frac{d'(s)}{d(s)} \;=\; \frac{z_d}{\hat{y}}e^{-\beta s} \quad . \tag{4.3}$$

It can be seen from (3.4) that $\hat{y}$ has to be positive, and we assume that z_d is positive. Recall now a well known special function $Ei(x)$, called *exponential integral*, defined by

$$Ei(x) \;:=\; -\int_{-x}^{\infty}\frac{e^{-t}}{t}dt \quad . \tag{4.4}$$

This is a well defined function except at $x = 0$. We are only interested in $x < 0$, where this function is continuous and decreases from 0 to $-\infty$. In other words,

$$Ei(-\infty) = 0 \qquad \text{and} \qquad Ei(0) = -\infty \quad .$$

We note that, for every $x < 0$:

$$Ei'(x) = \frac{e^x}{x} \qquad \text{and} \qquad Ei''(x) = -\frac{e^x}{x^2} < 0.$$

We can see that

$$-Ei'(-x) \;=\; e^{-x}/x \quad ,$$

so that integrating (4.3) we get

$$Ei(-d(s)) \;=\; -\frac{z_d}{\hat{y}\beta}e^{-\beta s} + C \quad . \tag{4.5}$$

We take C to be a non-positive constant. Since U_1 is not defined on negative numbers, we want the potential contract $d(s)$ to be positive. Thus, we consider the inverse function Ei^{-1} on the domain $(-\infty, 0)$, with values in $(-\infty, 0)$. We see that

$$d(s) \;=\; -Ei^{-1}\left(-\frac{z_d}{\hat{y}\beta}e^{-\beta s} + C\right) \quad . \tag{4.6}$$

This is a well defined function on $s \in (-\infty, \infty)$, continuous and increasing, with

$$d(-\infty) = 0 \qquad \text{and} \qquad d(\infty) \in (0, \infty] \quad .$$

If $C < 0$ then $d(\infty) < \infty$, if $C = 0$ then $d(\infty) = \infty$. We verify the remaining assumptions of Proposition 3.2 in the Appendix, showing that the contract $d(S_T)$ is optimal.

Example 4.4 (*Call Option Contract.*) We still assume that $\delta = D = q = 0$. We can use Proposition 3.2 to ask ourselves a reverse question: For which utilities is a given contract $d(S_T)$ optimal? In this example we show that using log utilities an option-like contract

$$d(s) = n(s - K)^+$$

can be optimal, for some $K > 0, n > 0$. We assume

$$S_0 > K \qquad \text{and} \qquad \beta > n \ \ .$$

Suppose that

$$U_1(x) = c\log(x)$$

for some $c > 0$. Then,

$$I_1(z) = c/z,$$

and we can see that the maximum in

$$\bar{U}_1^d(z) \ = \ \max_{s>0}\{U_1(d(s)) - sz\}$$

is $\hat{s} = c/z + K$, which means, as in proof of Proposition 3.1 in the Appendix, that the agent will act so that

$$S_T \ = \ \frac{c}{z_d Z_T} + K \tag{4.7}$$

for the value of z_d for which $E[Z_T S_T] = S_0$, which is equivalent to

$$\frac{c}{z_d} \ = \ S_0 - K \ \ . \tag{4.8}$$

Analyzing the ODE from Proposition 3.2, we see that we should try the utility of the form $U_2(x) = c_1 \log(x - c_2)$ for the principal. More precisely, let us assume that

$$U_2(x) = \begin{cases} \frac{cy}{z_d}(\beta - n)\log(x - \beta K) & \text{if } x > \beta K \\ -\infty & \text{if } x \le \beta K. \end{cases} \tag{4.9}$$

Here, $y > 0$ is a constant. Note that

$$I_2(z) \ = \ (\beta - n)\frac{cy}{z_d z} + \beta K \ \ .$$

Now, the principal gets his utility from

$$\beta S_T - n(S(T) - K)^+ \ = \ (\beta - n)\frac{c}{z_d Z_T} + \beta K \ \ ,$$

so we would like this expression to be equal to $I_2(\hat{y} Z_T)$, since this gives the maximum utility for the principal. This will be true if $y = \hat{y}$. Here, $\hat{y}$ has to satisfy the original principal's feasibility constraint (see (3.4))

$$\beta S_0 \ = \ E[Z_T I_1(\hat{z} Z_T) + Z_T I_2(\hat{y} Z_T)], \tag{4.10}$$

where $\hat{z}$ satisfies the original IR constraint (see (7.12) in the Appendix)

$$E\left[c\log\left(\frac{c}{\hat{z} Z_T}\right)\right] \ = \ R \ \ . \tag{4.11}$$

On the other hand, the IR constraint with the present contract is

$$E\left[U_1(n(S_T-K)^+)\right] \;=\; E\left[c\log\left(\frac{nc}{z_d Z_T}\right)\right] \;=\; R \quad .$$

From the last two equations we see that we need to have

$$n\hat{z} = z_d \quad ,$$

and if one of the equations is satisfied, the other will be, too. With $\hat{z} = z_d/n$, the condition (4.10) becomes

$$\beta S_0 \;=\; \beta K + \beta\frac{c}{z_d} \quad ,$$

which is true by (4.8).

Finally, we see that condition (4.11) becomes

$$R \;=\; E\left[c\log\left(\frac{n(S_0-K)}{Z_T}\right)\right] \;=\; c\left[\log\{n(S_0-K)\}+\alpha^2T/2\right] \quad . \tag{4.12}$$

To recap, this is what we have shown: Assuming $\delta = D = q = 0$, consider any values $c > 0$, $K < S_0$, $\beta > n$ such that (4.12) is satisfied. Then, if the agent and the principal have the utilities $U_1(x) = c\log(x)$, $U_2(x) = b\log(x - \beta K)$ respectively, for some constants $b, c > 0$, then the option-like contract $d(s) = n(s-K)^+$ is optimal. From (4.9) we can interpret βK as the lower bound on his wealth that the principal is willing to tolerate. We see that the higher this bound, the higher the strike price K will be.

5 More general models

In this section we formulate a very general model, and suggest a duality approach for solving the principal-agent problem in that context. We also present an example in which the problem solved in Ou-Yang (2003) is solved here using our approach. His approach uses Hamilton-Jacobi-Bellman partial differential equations, and is thus restricted to diffusion models, while our approach can be applied in general models, even though it may not always lead to explicit solutions. For simplicity, we assume that there is no dividend rate D and compensation rate q, although the analysis would be similar.

Consider the model in which the underlying process S controlled by the agent is of the form

$$dS_t \;=\; \delta a_t dt + \theta(t,S_t)dt + \sigma_t' d\bar{X}_t \quad .$$

Here, $\bar{X}$ is a $n-$dimensional RCLL semimartingale process on a given probability space. In addition a and σ are control processes. We assume that a is one-dimensional and σ is n-dimensional, and that both of them are adapted to a given filtration. Besides, $\theta = \theta(t, S_t)$ is a possibly random functional of (t, S_t), such that the above equation has a unique solution for constant values of a and σ.

The agent's problem is to maximize

$$E\left[U_1\left(P_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right)\right] \tag{5.1}$$

where P_T is the payoff paid by the principal, and $\bar{G}$ is a cost functional, possibly random.

The principal's problem is to maximize

$$E\left[U_2\left(\beta S_T - P_T\right)\right] \quad , \tag{5.2}$$

under the IR constraint that the agent's expected optimal utility is not less than R.

In order to describe a candidate solution, we introduce some notation. Let us denote by $\mathcal{D}_L$ the set of all adapted stochastic processes Z for which

$$E\left[Z_T\left(\beta S_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right)\right] \;\leq\; L(Z) \tag{5.3}$$

for some real valued functional $L(\cdot)$ acting on the process Z, such that $L(Z)$ is independent of a, σ, S, and such that

$$\forall \eta \in (0, \infty): \qquad L(\eta Z) = \eta L(Z). \tag{5.4}$$

We are denoting by Z the stochastic process, and by Z_t the random variable which is the value of the stochastic process Z at time t. For a given process $Z \in \mathcal{D}_L$, consider the optimization problem

$$H(Z) \;:=\; \inf_{Y \in \mathcal{D}_L} E\left[\tilde{U}_1(Y_T) + L(Y) - Y_T I_2(Z_T)\right]. \tag{5.5}$$

Here, for each $i \in \{1, 2\}$,

$$\tilde{U}_i(z) \;:=\; \max_x \{U_i(x) - xz\}. \tag{5.6}$$

Consider now those stochastic processes Z for which the infimum above is attained at some stochastic process $\hat{Y}(Z) =: F(Z)$, where we just defined a new functional F. We have the following main result of this section.

Theorem 5.1 *Fix a functional L as above. Suppose that the mapping $F(\cdot)$ has a "fixed point" $\hat{Z}$ in the sense that*

$$F(\hat{Z}) = \bar{c}\hat{Z} \tag{5.7}$$

for some positive constant $\bar{c}$. Suppose also that there exist admissible processes $\hat{a}, \hat{\sigma}$ such that

$$\beta S_T - I_2(\hat{Z}_T) - \int_0^T \bar{G}(t, \hat{a}_t, \hat{\sigma}_t, S_t)dt \;=\; I_1(\bar{c}\hat{Z}_T) \tag{5.8}$$

and

$$E\left[\hat{Z}_T\left(\beta S_T - \int_0^T \bar{G}(t, \hat{a}_t, \hat{\sigma}_t, S_t)dt\right)\right] \;=\; L(\hat{Z}) \quad . \tag{5.9}$$

Suppose also that the IR constraint

$$E[U_1(I_1(\bar{c}\hat{Z}_T))] \;=\; R \tag{5.10}$$

holds. Then, the first-best solution is the pair $(\hat{a}, \hat{\sigma})$ *and the contract*

$$\hat{P}_T \;=\; \beta S_T - I_2(\hat{Z}_T) \tag{5.11}$$

implements the first-best solution.

Proof: See the Appendix. $\diamond$

Remark 5.1 The existence of the optimal $\hat{Y}(Z)$ for the problem (5.5) and the existence of corresponding $\hat{a}, \hat{\sigma}$ has been studied in various papers in different models, in a context of dual problem to the problem of maximizing expected utility for an investor trading in financial markets, initially in Brownian Motion models, and recently in more general semimartingale models. Brownian Motion models where the number of Brownian Motions driving the model is higher than the dimension of the control vector σ were analyzed in He and Pearson (1991) and Karatzas et al (1991). The case in which the control vector σ is constrained to take values in a convex set was resolved in Cvitanić and Karatzas (1992). See also the book Karatzas and Shreve (1998). In those papers there is no cost in applying σ. The case corresponding to the cost function $\bar{G} = \bar{G}(\sigma, S)$ being nonlinear was studied in Cvitanić (1997) and Cuoco and Cvitanić (1998). Models more general than Brownian models have been studied in Kramkov and Schachermayer (1999), Cvitanić, Schachermayer and Wang (2001), Hugonnier and Kramkov (2002), Karatzas and Žitković (2002), among others. These are typically very hard problems, so that we expect that general results on the existence of the fixed point in the above theorem to be difficult to obtain. This is left for future research, while here we present examples in which the fixed point exists, and the approach works.

5.1 Examples

5.1.1 Example: Exponential Utility with volatility and size penalty

We illustrate here the power of the above approach by considering an example which was solved in Ou-Yang (2003) using Hamilton-Jacobi-Bellman partial differential equations. For ease of notation we assume that $\beta = 1$. The underlying process is driven by a d-dimensional Brownian Motion W:

$$dS_t \; = \; rS_t dt + \alpha' \sigma_t dt + \sigma_t' dW_t \quad ,$$

where $a'b$ denotes the inner product of two d-dimensional vectors a and b. We also consider a penalty of the type

$$\bar{G}(t, \sigma, s) \; = \; g(t, \sigma) + \gamma s \quad ,$$

that is, linear in the size S of the controlled process.

We will show that the contract (5.19) below is optimal. We want to find a candidate process $\hat{Z}$ corresponding to the optimal contract. Motivated by the results of Cvitanić (1997) and Cuoco and Cvitanić (1998), we consider an adapted vector processes $\lambda = \{\lambda_t; t \geq 0\}$ such that

$$E\left[\int_0^T \|\lambda_s\|^2 ds\right] < \infty \quad ,$$

and define the process Z^λ by

$$Z_t^\lambda \; := \; \exp\left\{-\int_0^t (\alpha + \lambda_s)' dW_s - \frac{1}{2}\int_0^t \|\alpha + \lambda_s\|^2 ds\right\} \quad ,$$

satisfying

$$dZ_t^\lambda = -Z_t^\lambda (\alpha + \lambda_t)' dW_t \quad , \quad Z_0^\lambda = 1 \quad .$$

Next, we want to get upper an bound of the form (5.3). Integration by parts implies

$$\begin{aligned} \int_0^T \gamma S_t dt \; &= \; \int_0^T \gamma e^{rt}(e^{-rt} S_t) dt \\ &= \; \frac{\gamma}{r}(e^{rT} - 1)e^{-rT} S_T - \int_0^T \frac{\gamma}{r}(1 - e^{-rt})[\alpha' \sigma_t dt + \sigma_t' dW_t] \quad . \end{aligned}$$

Moreover, using this and Itô's rule, we can see that

$$\begin{aligned} d\left(Z_t^\lambda \int_0^t [\gamma S_s + g(s, \sigma_s)] ds\right) \; &= \; d\left(Z_t^\lambda \frac{\gamma}{r}(e^{rt} - 1)e^{-rt} S_t\right) + Z_t^\lambda \left[\frac{\gamma}{r}(1 - e^{-rt})\lambda_t' \sigma_t + g(t, \sigma_t)\right] dt \\ &\quad + (\ldots) dW_t \quad . \end{aligned}$$

Also, by Itô's rule,

$$d\left(Z_t^\lambda e^{-rt} S_t\right) \; = \; -Z_t^\lambda e^{-rt} \lambda_t' \sigma_t dt + (\ldots) dW_t \quad .$$

Using the last two equations, and assuming that all $(\ldots)dW_t$ terms are martingales (not just local martingales), so that their expectation is zero, we get

$$E\left[Z_T^\lambda \left(S_T - \int_0^T (\gamma S_t + g(t, \sigma_t)) dt\right)\right]$$

$$= \left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right) E[Z_T^\lambda e^{-rT} S_T] - E\left[\int_0^T Z_t^\lambda \left[\frac{\gamma}{r}(1-e^{-rt})\lambda_t'\sigma_t + g(t,\sigma_t)\right] dt\right]$$
$$= \left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right) S_0 - E\left[\int_0^T Z_t^\lambda \left[\left(\frac{\gamma}{r} + (1-\frac{\gamma}{r})e^{r(T-t)}\right)\lambda_t'\sigma_t + g(t,\sigma_t)\right] dt\right] \quad . \tag{5.12}$$

Denote

$$f(t) \;=\; \frac{\gamma}{r} + (1-\frac{\gamma}{r})e^{r(T-t)} \quad .$$

Similarly as in Cuoco and Cvitanić (1998) and Cvitanić (1997), we define the dual function

$$\tilde{g}(t,\lambda) \;:=\; \max_\sigma\{-g(t,\sigma) - f(t)\sigma'\lambda\}$$

for those vectors λ for which this is well defined (not equal to infinity), which then make up the effective domain of $\tilde{g}$. We now consider only those vector processes $\lambda = \{\lambda_t; t \geq 0\}$ which take values in the effective domain of $\tilde{g}$. By the definition of the dual function and (5.12), we get

$$E\left[Z_T^\lambda\left(S_T - \int_0^T(\gamma S_t + g(t,\sigma_t))dt\right)\right]$$
$$\leq \; L(Z^\lambda) := \left(e^{rT} - \frac{\gamma}{r}(e^{rT}-1)\right) S_0 + E\left[\int_0^T Z_t^\lambda \tilde{g}(t,\lambda_t) dt\right] \quad . \tag{5.13}$$

This is an upper bound of the form (5.3).

Hence, for a given stochastic process ρ and given constants z, y, we want to do the following minimization:

$$H(Z^\rho) \;:=\; \inf_\lambda E\left[\tilde{U}_1(zZ_T^\lambda) + L(zZ^\lambda) - zZ_T^\lambda I_2(yZ_T^\rho)\right] \tag{5.14}$$

similarly as in (5.5). Assume now that the utilities are exponential and the cost function is quadratic:

$$U_i(x) = -\frac{1}{\gamma_i}e^{-\gamma_i x} \qquad \text{and} \qquad g(t,\sigma) = \frac{1}{2}\sigma' k_t \sigma \quad ,$$

for some matrix valued function k_t. Note that the solution for the problem of minimizing a quadratic form

$$y'x + \frac{1}{2}x'kx$$

over x is given by

$$\hat{x} \;=\; -\left(\frac{k+k'}{2}\right)^{-1} y \quad . \tag{5.15}$$

Thus the maximum in the definition of $\tilde{g}$ is attained for, suppressing dependence on t,

$$\hat{\sigma} \;=\; -f\left(\frac{k+k'}{2}\right)^{-1}\lambda \quad , \tag{5.16}$$

and we have

$$\tilde{g}(t,\lambda) \;=\; f^2\lambda'\left[\left(\frac{k+k'}{2}\right)^{-1} - \left(\frac{k+k'}{2}\right)^{-1}\frac{k}{2}\left(\frac{k+k'}{2}\right)^{-1}\right]\lambda \;\; .$$

In our case

$$\tilde{U}_i(z) = -\frac{1}{\gamma_i}z + \frac{1}{\gamma_i}z\log(z) \qquad \text{and} \qquad I_i(z) = -\frac{1}{\gamma_i}\log(z) \;\; .$$

Thus, assuming again that all local martingales are martingales, the dual problem (5.14) is equivalent to minimizing

$$E\left[\frac{1}{\gamma_1}zZ_T^\lambda\log(zZ_T^\lambda) + \frac{1}{\gamma_2}zZ_T^\lambda\log(yZ_T^\rho) + \int_0^T zZ_t^\lambda\tilde{g}(t,\lambda_t)dt\right] \;\; .$$

This, in turn, is equivalent to minimizing

$$\begin{aligned} &E\left[\int_0^T Z_t^\lambda\left(\frac{1}{2\gamma_1}\|\alpha+\lambda_s\|^2 - \frac{1}{\gamma_2}\left[\|\alpha+\rho_s\|^2/2 - (\alpha+\lambda_s)'(\alpha+\rho_s)\right]\right)ds\right] \\ &+ \;\; E\left[\int_0^T Z_t^\lambda\left(f_s^2\lambda_s'\left[\left(\frac{k_s+k_s'}{2}\right)^{-1} - \left(\frac{k_s+k_s'}{2}\right)^{-1}\frac{k_s}{2}\left(\frac{k_s+k_s'}{2}\right)^{-1}\right]\lambda_s\right)ds\right] \;\; . \end{aligned}$$

We conjecture now that the optimal λ is deterministic, if ρ is deterministic. That means we simply have to maximize the quadratic form in the integral above, and by (5.15), the optimal λ is given by (suppressing dependence on t)

$$-\lambda \;=\; \left[\frac{1}{\gamma_1}\mathbf{i} + f^2\left(\frac{k+k'}{2}\right)^{-1}\right]^{-1}\left(\frac{\alpha}{\gamma_1} + \frac{1}{\gamma_2}(\alpha+\rho)\right) \quad ,$$

where $\mathbf{i}$ is the identity matrix. It can now be verified that this is indeed the optimal $\lambda = \lambda(\rho)$ for a given deterministic process ρ, for example by checking that the Hamilton-Jacobi-Bellman equation is satisfied.

The fixed point is obtained by setting $\rho = \lambda$, which gives

$$\hat{\lambda} \;=\; -\Gamma\left[\Gamma\mathbf{i} + f^2\left(\frac{k+k'}{2}\right)^{-1}\right]^{-1}\alpha \tag{5.17}$$

where

$$\Gamma \;:=\; \frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2} \quad . \tag{5.18}$$

We would like to use Theorem 5.1 to claim that the contract

$$\hat{P}_T \;=\; S_T - I_2(\hat{y}Z_T^{\hat{\lambda}}) \;=\; S_T + \frac{1}{\gamma_2}\left[\log(\hat{y}) - \int_0^T\|\alpha+\hat{\lambda}_s\|^2/2ds - \int_0^T(\alpha+\hat{\lambda}_s)'dW_s\right] \tag{5.19}$$

is optimal for an appropriate choice of $\hat{y}$, to be determined below. It can be checked that this is the same contract as in Ou-Yang (2003), except that the term dW_t there is expressed in terms of a different process, having the interpretation of stock prices. Also, in his framework, the process S has the interpretation of the value of a managed fund.

We would like to check that the conditions (5.8) and (5.9) are satisfied with the stochastic processes Z and Y defined by $Z_t = \hat{z} Z_t^{\hat{\lambda}}$ and $Y_t = \hat{y} Z_t^{\hat{\lambda}}$, for some values of $\hat{z}, \hat{y}$. We choose $\hat{z}$ so that the IR constraint (5.10) is satisfied. In order to satisfy (5.8) we need to have

$$S_T = I_1(\hat{z} Z_T^{\hat{\lambda}}) + I_2(\hat{y} Z_T^{\hat{\lambda}}) + \int_0^T [g(s, \sigma_s) + \gamma S_s] ds \quad ,$$

or, after integration by parts,

$$\begin{aligned}\left(e^{rT} - \frac{\gamma}{r}(e^{rT} - 1)\right) e^{-rT} S_T &= I_1(\hat{z} Z_T^{\hat{\lambda}}) + I_2(\hat{y} Z_T^{\hat{\lambda}}) \\ &\quad + \int_0^T \left[g(s, \sigma_s) - \frac{\gamma}{r}(1 - e^{-rs})\alpha' \sigma_s'\right] ds - \int_0^T \frac{\gamma}{r}(1 - e^{-rs})\sigma_s' dW_s \quad .\end{aligned}$$

Using

$$e^{-rT} S_T = S_0 + \int_0^T e^{-rs} \alpha' \sigma_s ds + \int_0^T e^{-rs} \sigma_s' dW_s$$

and substituting for I_1 and I_2, we need to have

$$\begin{aligned}&\left(e^{rT} - \frac{\gamma}{r}(e^{rT} - 1)\right)\left(S_0 + \int_0^T e^{-rs}\alpha'\sigma_s ds + \int_0^T e^{-rs}\sigma_s' dW_s\right) \\ &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) + \frac{\Gamma}{2}\int_0^T \|\alpha + \hat{\lambda}_s\|^2 ds \qquad (5.20)\\ &\quad +\Gamma \int_0^T (\alpha + \lambda_s)' dW_s + \int_0^T \left[g(s, \sigma_s) - \frac{\gamma}{r}(1 - e^{-rs})\alpha'\sigma_s'\right] ds - \int_0^T \frac{\gamma}{r}(1 - e^{-rs})\sigma_s' dW_s \quad .\end{aligned}$$

By (5.16), we conjecture that we have to take σ to be

$$\hat{\sigma} = -f\left(\frac{k + k'}{2}\right)^{-1} \hat{\lambda} = f\left[\frac{f^2}{\Gamma}\mathbf{i} + \frac{k + k'}{2}\right]^{-1} \alpha \quad .$$

Indeed, it can now be verified, using the fact that

$$(\alpha + \hat{\lambda})\Gamma = f\hat{\sigma} \quad ,$$

that if we choose this value for σ, then in the equation (5.20) the dW integrals cancel out. In order for the remaining terms to satisfy equation (5.20), we need to have

$$\begin{aligned}0 &= \left(e^{rT} - \frac{\gamma}{r}(e^{rT} - 1)\right) S_0 + \int_0^T \left[f(s)\alpha'\hat{\sigma}_s - \frac{1}{2\Gamma}\int_0^T f^2(s)\|\hat{\sigma}_s\|^2 - \frac{1}{2}\hat{\sigma}_s' k_s \hat{\sigma}_s\right] ds \\ &\quad + \frac{1}{\gamma_1}\log(\hat{z}) + \frac{1}{\gamma_2}\log(\hat{y}) \quad .\end{aligned}$$

This is an equation in $\hat{y}$ that can be solved. It is now easily verified that (5.9) is also satisfied. Thus, by Theorem 5.1 the contract (5.19) is indeed optimal.

5.1.2 Example: Original Holmstrom-Milgrom problem

In this example we show that if we restrict the principal to observe only the controlled process S, but not the driving Brownian Motion W, then his utility may be strictly lower than if he can offer contracts based on both S and W. We recall a one-dimensional case of the Holmstrom and Milgrom (1987) setting, where only the drift is controlled,

$$dS_t \;=\; a_t dt + dW_t$$

where W is a one-dimensional Brownian Motion. The computations are similar, but simpler than in the previous example, and we omit the details. We could also do the multi-dimensional case, as in that example. We set $\beta = 1$ for notational simplicity. We denote similarly as in the previous example,

$$Z_t^\lambda \;=\; \exp\left\{-\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds\right\} \quad .$$

We assume that the cost function is $\bar{G}(a) = a^2/2$. By Itô's rule,

$$d\left(Z_t^\lambda\left(S_t - \int_0^t \frac{1}{2}a_s^2 ds\right)\right) \;=\; Z_t^\lambda\left[a_t - a_t^2/2 - \lambda_t\right]dt + (\ldots)dW_t \quad .$$

Using this and the fact that $a - a^2/2$ is maximized at $\hat{a} = 1$, and assuming that all $(\ldots)dW_t$ terms are martingales (not just local martingales), so that their expectation is zero, we get

$$E\left[Z_T^\lambda\left(S_T - \int_0^T \frac{1}{2}a_s^2 ds\right)\right] \;\le\; \frac{T}{2} + S_0 - E\left[\int_0^T Z_t^\lambda \lambda_t dt\right] \quad .$$

This is an upper bound of the form (5.3). Hence, for a given stochastic process ρ and given constants z, y, we want to do the following minimization:

$$G(Z^\rho) \;:=\; \inf_\lambda E\left[\tilde{U}_1(zZ_T^\lambda) - z\int_0^T Z_t^\lambda \lambda_t dt - zZ_T^\lambda I_2(yZ_T^\rho)\right] \tag{5.21}$$

similarly as in (5.5). Assume now that the utilities are exponential,

$$U_i(x) \;=\; -\frac{1}{\gamma_i}e^{-\gamma_i x} \quad .$$

We have then

$$\tilde{U}_i(z) = -\frac{1}{\gamma_i}z + \frac{1}{\gamma_i}z\log(z) \qquad \text{and} \qquad I_i(z) = -\frac{1}{\gamma_i}\log(z) \quad .$$

Thus, assuming again that all local martingales are martingales, the dual problem (5.21) is equivalent to minimizing

$$E\left[\frac{1}{\gamma_1}zZ_T^\lambda\log(zZ_T^\lambda) + \frac{1}{\gamma_2}zZ_T^\lambda\log(yZ_T^\rho) - z\int_0^T Z_t^\lambda \lambda_t dt\right] \quad .$$

This, in turn, is equivalent to minimizing

$$E\left[\int_0^T Z_t^\lambda\left(\frac{1}{2\gamma_1}\lambda_s^2 - \frac{1}{\gamma_2}\left[\frac{\rho_s^2}{2} - \lambda_s\rho_s\right]\right)ds\right] - E\left[\int_0^T Z_t^\lambda \lambda_t dt\right] \quad .$$

We again conjecture that the optimal λ is deterministic, if ρ is deterministic. That means we simply have to maximize the quadratic function in the integral above, and the optimal λ is given from (suppressing dependence on t)

$$\frac{\lambda}{\gamma_1} + \frac{\rho}{\gamma_2} = 1 \quad .$$

It can now be verified that this is indeed the optimal $\lambda = \lambda(\rho)$ for a given deterministic process ρ, for example by checking that the Hamilton-Jacobi-Bellman equation is satisfied.

The fixed point is obtained by setting $\rho = \lambda$, which gives

$$\hat{\lambda} = \frac{1}{\Gamma} \tag{5.22}$$

where

$$\Gamma := \frac{\gamma_1 + \gamma_2}{\gamma_1\gamma_2} \quad . \tag{5.23}$$

By Theorem 5.1, the contract

$$\hat{P}_T = S_T - I_2(\hat{y}Z_T^{\hat{\lambda}}) = S_T + \frac{1}{\gamma_2}\left[\log(\hat{y}) - \int_0^T \frac{1}{2}\hat{\lambda}_s^2 ds - \int_0^T \hat{\lambda}_s dW_s\right] \tag{5.24}$$

is optimal for an appropriate choice of $\hat{y}$, to be determined below. We have to check that the conditions (5.8) and (5.9) are satisfied with the stochastic processes $\hat{Z}$ and $\hat{Y}$ defined by $\hat{Z}_t = \hat{z}Z_t^{\hat{\lambda}}$ and $\hat{Y}_t = \hat{y}Z_t^{\hat{\lambda}}$, for some values of $\hat{z}, \hat{y}$. We choose $\hat{z}$ so that the IR constraint (5.10) is satisfied, that is

$$\hat{z} = -\gamma_1 R \quad . \tag{5.25}$$

Since we are taking $\hat{a} \equiv 1$, in order to satisfy (5.8) we need to have

$$\begin{aligned} S_T &= I_1(\hat{z}Z_T^{\hat{\lambda}}) + I_2(\hat{y}Z_T^{\hat{\lambda}}) + T/2 \\ &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_1}\log(Z_T^{\hat{\lambda}}) - \frac{1}{\gamma_2}\log(\hat{y}) - \frac{1}{\gamma_2}\log(Z_T^{\hat{\lambda}}) + \frac{T}{2}. \end{aligned}$$

Then,

$$\begin{aligned} S_0 + T + W_T &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) - \Gamma\log(Z_T^{\hat{\lambda}}) + \frac{T}{2} \\ &= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) + W_T + \frac{1}{2}\frac{1}{\Gamma}T + \frac{T}{2}, \end{aligned}$$

or equivalently

$$S_0 + \frac{T}{2} - \hat{\lambda}\frac{T}{2} = -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) \quad . \tag{5.26}$$

It is now easily verified that (5.9) is also satisfied. The maximum utility is given by

$$E\left[U_2\left(S_T - I_2(\hat{y}Z_T^{\hat{\lambda}})\right)\right] = -\frac{\hat{y}}{\gamma_2} \quad , \tag{5.27}$$

where $\hat{y}$ is determined from (5.26), with $\hat{z}$ determined from (5.25).

Let us now compare this to the utility obtained from the optimal contract among those contracts which depend only on S, not on W. By Holmstrom and Milgrom (1987) (see also Schättler and Sung 1993), the optimal contract of such a form is linear, and given by

$$f(S_T) = c + bS_T \quad ,$$

where

$$b = \frac{1+\gamma_2}{1+\gamma_1+\gamma_2} \quad ,$$

and c is chosen so that the IR constraint is satisfied. In this case the optimal control is $\hat{a} = b$ and the IR constraint gives

$$c = -\frac{1}{\gamma_1}\log(-\gamma_1 R) - bS_0 + \frac{b^2 T}{2}(\gamma_1 - 1) \quad . \tag{5.28}$$

The maximum principal's utility with this contract is

$$E\left[U_2(S_T - (c + bS_T))\right] = -\frac{1}{\gamma_2}\exp\left\{-\gamma_2\left[-c + (1-b)\left(S_0 + bT - \frac{1}{2}\gamma_2(1-b)T\right)\right]\right\} \quad .$$

Comparing this to the utility from (5.27), we can check that the difference of the utility from the generally optimal contract $\hat{P}_T = S_T - I_2(\hat{y}Z_T^{\hat{\lambda}})$ and the utility from the optimal S-based contract is strictly positive.

We see that in this example the first best contract $\hat{P}_T$ is not implementable with S-based contracts.

5.1.3 Example: Volatility constraints

Using our method, it can be shown that in the model

$$dS_t = \alpha\sigma_t dt + \sigma_t W_t,$$

where W is a one-dimensional Brownian Motion, if the principal is risk-neutral, $U_2(x) = x$, and the agent has no penalty on volatility, then the principal can attain infinite utility by forcing the agent to apply infinite volatility, $\hat{\sigma} \equiv \infty$. This may not be true if there are constraints on volatility, of the type $\sigma_t \in K$ for all t, where K is a closed and convex set. We could still use the approach of Example 5.1.2, with

$$\tilde{g}(t,\lambda) = \max_{\sigma\in K}\{-g(t,\sigma) - f(t)\sigma'\lambda\} \quad . \tag{5.29}$$

In the one dimensional case, we consider the constraint

$$b \le \sigma_t \le c$$

for some constants b, c (the example can be extended to time-dependent bounds b_t, c_t). For simplicity, we assume that the cost function is zero.

Since our method cannot deal directly with the risk-neutral utility function $U(x) = x$, we assume that the utility functions are exponential:

$$U_i(x) = \frac{1}{\gamma_i} - \frac{1}{\gamma_i} e^{-\gamma_i x} \quad .$$

Note that if we let γ_i go to zero, then $U_i(x)$ goes to x, and we get the risk-neutral case in the limit. We can now verify, similarly as in the previous examples, that the optimal volatility $\hat{\sigma}$ is given by

$$\hat{\sigma} = \begin{cases} b & ; \quad \text{if } \Gamma\alpha < b \\ c & ; \quad \text{if } \Gamma\alpha > c \\ \Gamma\alpha & ; \quad \text{if } b \le \Gamma\alpha \le c \end{cases}$$

where Γ is defined in (5.18). Finally, note that if either the principal or the agent is risk-neutral, that is, if either γ_1 or γ_2 tends to zero, then Γ tends to infinity, which implies that

$$\hat{\sigma} = c \quad ,$$

that is, the agent will optimally choose the maximum possible volatility.

6 Conclusions

In this paper we consider continuous-time principal-agent problems in which the agent can control both the diffusion and the drift term of the underlying process, and the principal can observe the agent's actions. In the case the agent does not control the drift independently of the diffusion term, and does not face any cost for it, the optimal contract is (ex-post) a function of the terminal value of the controlled process, and this function may be nonlinear if the agent and the principal have different utility functions. Even when the contract is ex-post linear, if the agent controls the drift separately from the diffusion term, the optimal contract may be path dependent ex-ante. If the agent does not control the drift independently of the diffusion term, the optimal contract can be offered as a function of the terminal value if a certain differential equation has a solution. Call option-type contracts are optimal for a specific choice of utility functions. The contract is linear when the agent and the principal both have exponential utility functions, or if they have the same power utility functions.

When it is costly for the agent to control the diffusion term, the optimal contract is obtained by considering a fixed point of a map between the principal's optimization problem and the agent's optimization problem.

There are several directions in which we could extend this work. Here, the agent and the principal have the same information, and the solution is the first-best. It would be of considerable interest to study the problem with asymmetric or incomplete information. Another, more realistic cases could be considered, such as a possibility for the agent to cash in the contract at a random time, or the case when the time horizon is also a part of the contract. We leave these problems for future research.

7 Appendix

Proof of Theorem 3.1: The following argument is similar to standard arguments in modern portfolio optimization literature, and due originally, in a somewhat different form, to Pliska (1986), Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987) (see, for example, Karatzas & Shreve (1998) for a general theory). For a heuristic derivation, see the proof of Proposition 3.1 below.

Define the dual function

$$U_2^*(y,\lambda) \;:=\; \max_{s,p}\{U_2(\beta s - p) + \lambda U_1(p) - y\beta s\} \tag{7.1}$$

for those values of $\lambda > 0$ and y for which this is finite, constituting the effective domain $\tilde{D}_2$ of U_2^*. Similarly, define the dual function

$$V_2^*(\bar{y},\lambda) \;:=\; \max_{D,q}\{V_2(\kappa D - q) + \lambda V_1(q) - \bar{y}\kappa D\}\,. \tag{7.2}$$

We assume, for simplicity, that V_2^* and U_2^* are defined on the same domain $\tilde{D}_2$.

We show in the lemma below that the maximums are attained at

$$\beta\hat{s} - \hat{p} = I_2^U(y)\;;\quad \hat{p} = I_1^U(y/\lambda)\quad , \tag{7.3}$$

$$\kappa\hat{D} - \hat{q} = I_2^V(\bar{y})\;;\quad \hat{q} = I_1^V(\bar{y}/\lambda)\quad . \tag{7.4}$$

Note that the maximum of $\beta\delta a - G(a)$ is attained at

$$\hat{a} \;\equiv\; J(\delta\beta)\quad . \tag{7.5}$$

This means, from (2.1), that

$$\beta S_T - \int_0^T G(a_t)dt \;\leq\; \beta S_T^\beta - TG(J(\delta\beta))\quad ,$$

where S^β indicates the process S for which $a \equiv J(\delta\beta)$.

Set now $\beta s = \beta S_T - \int_0^T G(a_t)dt$ and $p = P_T - \int_0^T G(a_t)dt$ in the definition of U_2^*. All of this, together with the IR constraint, gives us

$$E\left[U_2(\beta S_T - P_T) + \int_0^T V_2(\kappa D_s - q_s)ds\right] \tag{7.6}$$
$$\leq E\left[U_2^*(yZ_T,\lambda) + \int_0^T V_2^*\left(\frac{\beta}{\kappa}yZ_s,\lambda\right)ds\right] + yE\left[Z_T\left(\beta S_T^\beta - TG(J(\delta\beta))\right) + \beta\int_0^T Z_s D_s ds\right]$$
$$-\lambda R \quad ,$$

where R is the reservation utility for the agent and y is a constant. Also note that from the fact that the process M of (3.3) is a martingale, we have $E[M_T] = S_0$ and so we get the upper bound for the principal's problem:

$$E\left[U_2(\beta S_T - P_T) + \int_0^T V_2(\kappa D_s - q_s)ds\right] \tag{7.7}$$
$$\leq E\left[U_2^*(yZ_T,\lambda) + \int_0^T V_2^*\left(\frac{\beta}{\kappa}yZ_s,\lambda\right)ds\right] + y\left\{\beta S_0 + T\beta\delta J(\delta\beta) - TG(J(\delta\beta))\right\} - \lambda R \quad .$$

The upper bound is attained if (7.5) is satisfied, and, by (7.3), (7.4), if

$$P_T = \beta S_T - I_2^U(yZ_T) \quad , \tag{7.8}$$

$$P_T - TG(J(\delta\beta)) = I_1^U(yZ_T/\lambda) \quad , \tag{7.9}$$

$$q_t = \kappa D_t - I_2^V\left(\frac{\beta}{\kappa}yZ_t\right) \quad , \tag{7.10}$$

$$q_t = I_1^V\left(\frac{\beta}{\kappa\lambda}yZ_t\right) \quad , \tag{7.11}$$

if the IR constraint is satisfied as equality:

$$R = E\left[U_1\left(I_1^U(yZ_T/\lambda)\right) + \int_0^T V_1\left(I_1^V\left(\frac{\beta}{\kappa\lambda}yZ_t\right)\right)dt\right] \quad , \tag{7.12}$$

and if the martingale property $E[M_T] = S_0$ holds, or equivalently,

$$E\left[Z_T S_T^\beta + \int_0^T Z_s D_s ds\right] = S_0 + T\delta J(\delta\beta) \quad . \tag{7.13}$$

For a as in (7.5), equations (7.8)-(7.11) and (7.13) imply that the number $y = \hat{y}$ has to be chosen so that the *principal's feasibility constraint*

$$\beta(S_0 + T\delta J(\delta\beta)) \tag{7.14}$$
$$= E\left[Z_T\left\{I_1^U(\hat{y}Z_T/\lambda) + I_2^U(\hat{y}Z_T) + TG(J(\delta\beta))\right\} + \int_0^T \left\{Z_s\frac{\beta}{\kappa}\left[I_1^V(\frac{\beta}{\kappa\lambda}\hat{y}Z_s) + I_2^V(\frac{\beta}{\kappa}\hat{y}Z_s)\right]\right\}ds\right]$$

is satisfied. If we set $\lambda = \hat{z}/\hat{y}$, such $\hat{y}$ exists by Assumption 3.2, and the only remaining thing we have to check is whether there exists a process σ so that at the final time we have

$$\beta S_T \;=\; \beta \hat{S}_T \;=\; I_1^U(\hat{y}Z_T/\lambda) + I_2^U(\hat{y}Z_T) + TG(J(\delta\beta)), \tag{7.15}$$

with $\lambda = \hat{z}/\hat{y}$. Here we use the Martingale Representation Theorem, which says that we can write, for any given $\mathcal{F}_T-$measurable random variable M_T,

$$M_t \;:=\; E_t[M_T] \;=\; E[M_T] + \int_0^t \varphi_s^M dW_s$$

for some adapted process φ^M, where E_t denotes expectation conditional on the information available up to time t. We want to have, with $\hat{S}, \hat{a}, \hat{D}$ as in the statement of the theorem,

$$M_T \;=\; \hat{S}_T Z_T - \delta \int_0^T Z_s \hat{a}_s ds + \int_0^t Z_s \hat{D}_s ds$$

and we see from (3.3) that this is possible if we choose $\sigma = \hat{\sigma}$ so that

$$(\hat{\sigma}_t - \alpha S_t) Z_t \;=\; \varphi_t^M \quad .$$

Such choice of σ will then, indeed, result in (7.15).

◇

Lemma 7.1 *The values $\hat{s}, \hat{p}, \hat{D}, \hat{q}$ from* (7.3), (7.4) *are optimal for the maximization in the definition of U_2^*, V_2^*.*

Proof: We only show the case for U_2^*. Values $\hat{s}, \hat{p}$ are determined so that they make the partial derivatives equal to zero. Thus, it only remains to show that the Hessian of the problem is a negative definite matrix. Let us introduce the change of variables

$$\tilde{s} = \beta s \quad .$$

Denote

$$f(\tilde{s}, p) \;=\; U_2(\tilde{s} - p) - y\tilde{s} + \lambda U_1(p).$$

We compute the second derivatives as follows:

$$\begin{aligned}
\frac{\partial^2 f(\tilde{s}, p)}{\partial \tilde{s}^2} &= U_2''(\tilde{s} - p), \\
\frac{\partial^2 f(\tilde{s}, p)}{\partial p^2} &= U_2''(\tilde{s} - p) + \lambda U_1''(p) \quad , \\
\frac{\partial^2 f(\tilde{s}, p)}{\partial p \partial \tilde{s}} &= -U_2''(\tilde{s} - p).
\end{aligned}$$

Then we find that, for a given vector $(a,b)' \neq (0,0)'$ and the Hessian matrix $H(\tilde{s},p)$, we have

$$(a,b)H(\tilde{s},p)(a,b)' \;=\; U_2''(\tilde{s}-p)(a-b)^2 + b^2\lambda U_1''(p).$$

Since $\lambda > 0$, and since U_2, U_1 are strictly concave, the right-hand side is negative.

$\diamond$

Proof of Proposition 3.1: We first develop some heuristics for finding the optimal strategy of the agent, while a rigorous proof is given after that. We can consider the fact that the process M is a martingale, and in particular that $E[M_T] = S_0$, to be a constraint on the agent's problem. Thus, assuming that there is an optimal process $\hat{a}$ and fixing it, the has to find controls σ, D that maximize

$$E\left[U_1\left(\beta S_T - I_2^U(\hat{y}Z_T) - \int_0^T G(\hat{a}_s)ds\right) + \int_0^T V_1\left(\kappa D_s - I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_s\right)\right)ds\right]$$
$$-\beta z_A\left(E\left[Z_T S_T - \delta\int_0^T Z_s\hat{a}_s ds + \int_0^T Z_s D_s ds\right] - S_0\right)$$

where z_A is a Lagrange multiplier. Taking derivatives with respect to S_T, D_s and setting them equal to zero (and neglecting the expectation) in the previous maximization problem, we conjecture that it is optimal to choose the controls $\hat{\sigma}, \hat{D}$ so that

$$\beta S_T - I_2^U(\hat{y}Z_T) - \int_0^T G(\hat{a}_s)ds \;=\; I_1^U(z_A Z_T) \tag{7.16}$$

and

$$\kappa\hat{D}_t - I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_t\right) \;=\; I_1^V\left(\frac{\beta}{\kappa}z_A Z_t\right). \tag{7.17}$$

If we substitute this into the martingale property $E[M_T] = S_0$, we get that the number z_A has to satisfy the *agent's feasibility constraint*

$$\beta S_0 = E\left[Z_T\left\{I_1^U(z_A Z_T) + I_2^U(\hat{y}Z_T) + \int_0^T G(\hat{a}_s)ds\right\} - \delta\beta\int_0^T Z_s\hat{a}_s ds + \beta\int_0^T Z_s\hat{D}_s ds\right] \;. \tag{7.18}$$

In order to provide a rigorous proof of (7.16), (7.17), we need to introduce *dual functions* $\tilde{U}_1$, $\tilde{V}_1$ of U_1, V_1 defined by, for $f = U_1, V_1$,

$$\tilde{f}(z) \;:=\; \max_x\{f(x) - xz\} \;\;.$$

The domain $\tilde{D}_1$ of $\tilde{U}_1$ consists of the values of z for which $\tilde{U}_1(z) < \infty$. We assume for simplicity that $\tilde{V}_1$ has the same domain. Note that performing the maximization in $\tilde{f}$, we get that the optimal x is given by

$$x = I_1^f(z) \;\;, \tag{7.19}$$

where I_1^f is the inverse function of f'. As for the choice of a, we see that we can write

$$d\left(\beta S_t - \int_0^t G(a_s)ds\right) \;=\; \beta(\alpha\sigma_t - D_t)dt + \beta\sigma_t dW_t + \left[\delta\beta a_t - G(a_t)\right]dt. \tag{7.20}$$

If $\delta = 0$ the agent can set $a \equiv 0$. Let $\delta > 0$. Since U_1 is increasing, it is optimal to maximize the term $[\delta\beta a_t - G(a_t)]$ in (7.20), which gives the optimal value

$$\hat{a}_t = J(\delta\beta) \quad . \tag{7.21}$$

Next, by the definition of the dual functions we have

$$\begin{aligned} & E\left[U_1\left(\beta S_T - I_2^U(\hat{y}Z_T) - \int_0^T G(\hat{a}_s)ds\right) + \int_0^T V_1\left(\kappa D_s - I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_s\right)\right)ds\right] \qquad (7.22)\\ \le\;& E\left[\tilde{U}_1(z_A Z_T) + \int_0^T \tilde{V}_1\left(\frac{\beta}{\kappa}z_A Z_s\right)ds\right] \\ & + z_A E\left[Z_T\left(\beta S_T - I_2^U(\hat{y}Z_T) - \int_0^T G(\hat{a}_s)ds\right) + \int_0^T Z_s\left[\beta D_s - \frac{\beta}{\kappa}I_2^V\left(\frac{\beta}{\kappa}\hat{y}Z_s\right)\right]ds\right]. \end{aligned}$$

From $E[M_T] = S_0$ we get that, assuming the agent uses $\hat{a}$,

$$E\left[S_T Z_T + \int_0^T Z_s\hat{D}_s ds\right] \;=\; S_0 + \delta E\left[\int_0^T Z_s\hat{a}_s ds\right],$$

a quantity that does not depend on the choice of σ and D. Thus, from (7.22) we get an upper bound on the value of the agent's optimization problem. By (7.19), this upper bound is attained if (7.16) and (7.17) are satisfied, where z_A is chosen so that (7.18) is satisfied. With this, and comparing the feasibility constraints (7.14) and (7.18), we see that we need to take $z_A = \hat{y}/\lambda$, and thus, by choosing exactly the controls $\hat{a}, \hat{\sigma}, \hat{D}$ which are first-best, the agent will attain the upper bound for her utility.

◇

Proof of Proposition 3.2: Given such a contract it can be seen similarly as in the proof of Proposition 3.1 that the agent will optimally act so that the process S_t satisfies

$$Z_t S_t \;:=\; E_t[Z_T s(z_d Z_T)] \quad . \tag{7.23}$$

and in particular

$$S_T = s(z_d Z_T)$$

where the function $s(z)$ is determined from (3.14). From (3.14) and (3.12), we get

$$U_2'(\beta S_T - d(S_T)) \;=\; \hat{y}Z_T$$

or

$$\beta S_T - d(S_T) \ = \ I_2(\hat{y} Z_T).$$

But then the principal's utility is equal to

$$E[U_2(\beta S_T - P_T)] \ = \ E[U_2(I_2(\hat{y} Z_T))],$$

which is optimal.

◇

Proof of Corollary 3.1: Inspecting the proof of Theorem 3.1, but with $\delta = D = q = 0$, we see that any contract satisfying (3.16) attains the principal's upper bound, and is thus optimal. Moreover, that proof also shows that this equality has to be satisfied for the principal to attain his maximum utility.

◇

Proof of Theorem 3.2: Suppose that the contract is offered in the form $P_T = f(S_T) = I_1(\hat{z} h(S_T))$, assumed to be linear. Given this contract, similarly as in the proof of Proposition 3.1, we can see that the agent would choose $\sigma = \tilde{\sigma}$ so that

$$P_T = I_1(z^f Z_T)$$

for some z^f. Denote by $\tilde{S}$ the corresponding process S. We also have $P_T = I_1(\hat{z} h(\tilde{S}_T))$, thus

$$z^f Z_T = \hat{z} h(\tilde{S}_T). \tag{7.24}$$

Note that $\tilde{S}_T = f^{-1}(I_1(z^f Z_T))$, and by the martingale property,

$$S_0 \ = \ E[Z_T \tilde{S}_T] \ = \ E\left[Z_T f^{-1}(I_1(z^f Z_T))\right].$$

This means that $z^f = z^*$, the unique solution to the above equation. Denote now by $\hat{S}$ the process S corresponding to some optimal contract. From Corollary 3.1, any such contract satisfies $\beta \hat{S}_T = I_1(\hat{z} Z_T) + I_2(\hat{y} Z_T)$, which means that $\hat{S}_T = f^{-1}(I_1(\hat{z} Z_T))$. By the martingale property again, we have

$$S_0 \ = \ E[Z_T \hat{S}_T] \ = \ E\left[Z_T f^{-1}(I_1(\hat{z} Z_T))\right].$$

This means that also $\hat{z} = z^*$, thus $\hat{z} = z^f$, and from (7.24), $h(\tilde{S}_T) = Z_T$. Hence, we get

$$\beta \tilde{S}_T - P_T \ = \ \beta \tilde{S}_T - I_1(\hat{z} Z_T) = I_2(\hat{y} h(\tilde{S}_T)).$$

Therefore, the principal's utility when offering the contract $f(S_T)$ is

$$E\left[U_2(I_2(\hat{y} h(\tilde{S}_T)))\right] \ = \ E\left[U_2(I_2(\hat{y} Z_T))\right],$$

which is optimal.

$\diamond$

Computations for Example 4.2: Similarly to the proof of Proposition 3.1, the agent will choose the control $a = \tilde{a}$ given by

$$\tilde{a}_t = J(\delta b).$$

Using the duality method of that proof, we can see that he will choose $\sigma = \tilde{\sigma}$ so that

$$c + bS_T - TG(J(\delta b)) \;=\; -\frac{1}{\gamma_1}\log(\tilde{z}Z_T) \tag{7.25}$$

where $\tilde{z}$ is determined from the agent's feasibility (martingale) condition

$$c + bS_0 - TG(J(\delta b)) + T\delta b J(\delta b) \;=\; -\frac{1}{\gamma_1}(\log(\tilde{z}) + \alpha^2 T/2). \tag{7.26}$$

Here, we use the fact that

$$E\left[Z_T \log(Z_T)\right] \;=\; E\left[Z_T(-\alpha^2 T/2 - \alpha W_T)\right] \;=\; -\alpha^2 T/2 - \alpha E[Z_T W_T] \;=\; \alpha^2 T/2. \tag{7.27}$$

Now, the utility of the principal is, substituting S_T from (7.25),

$$\begin{aligned} E\left[U_2(\beta S_T - P_T)\right] &= E\left[U_2((\beta - b)S_T - c)\right] \\ &= E\left[U_2\left(\left(\frac{\beta}{b} - 1\right)\left[TG(J(\delta b)) - \frac{1}{\gamma_1}\log(\tilde{z}Z_T)\right] - c\frac{\beta}{b}\right)\right] \end{aligned} \tag{7.28}$$

Substituting for the utility function and the values a, b from (4.1), we get

$$\begin{aligned} E[U_2(\beta S_T - P_T)] &= E\Big[U_2\Big(\left(\frac{\gamma_1}{\gamma_2}\right)\left[TG(J(\delta b)) - \frac{1}{\gamma_1}\log(\tilde{z}Z_T)\right] \\ &\qquad - \left(\frac{\gamma_1}{\gamma_2}\right)\left[TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{y}/\hat{z})\right]\Big)\Big] \\ &= E\left[U_2\left(\left(\frac{\gamma_1}{\gamma_2}\right)\left[TG(J(\delta b)) - TG(J(\delta\beta))\right] - \frac{1}{\gamma_2}\log\left(\frac{\tilde{z}\hat{y}}{\hat{z}}Z_T\right)\right)\right] \\ &= -\frac{\hat{y}}{\gamma_2}\exp\left\{-\gamma_1\left[TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}/\tilde{z})\right]\right\}. \end{aligned} \tag{7.29}$$

On the other hand, we know that the principal's utility given the optimal contract $\hat{P} = \beta S_T - I_2(\hat{y}Z_T)$ is

$$E[U_2(\beta S_T - \hat{P}_T)] \;=\; E[U_2(I_2(\hat{y}Z_T))] \;=\; -\frac{\hat{y}}{\gamma_2}.$$

Thus, in order for the contract $f(S_T)$ to be optimal, the exponent in (7.29) should be equal to zero. In order to compute this exponent, we substitute $\tilde{z}$ from (7.26), we substitute b, c

from (4.1), and we also use the principal's feasibility (martingale) condition (7.14) (with $\lambda = \hat{y}/\hat{z}$) for the optimal contract:

$$\begin{aligned}
\beta S_0 &= -\frac{1}{\gamma_1} E\left[Z_T \log(\hat{z} Z_T)\right] - \frac{1}{\gamma_2} E\left[Z_T \log(\hat{y} Z_T)\right] + TG(J(\delta\beta)) - T\delta\beta J(\delta\beta) \\
&= -\frac{1}{\gamma_1}\log(\hat{z}) - \frac{1}{\gamma_2}\log(\hat{y}) + TG(J(\delta\beta)) - T\delta\beta J(\delta\beta) - \frac{\alpha^2 T}{2}(\gamma_1^{-1} + \gamma_2^{-1}). \quad (7.30)
\end{aligned}$$

Doing that we obtain from (7.30)

$$\begin{aligned}
& TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}/\tilde{z}) \\
= \;& TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}) + c + bS_0 - TG(J(\delta b)) + T\delta b J(\delta b) + \frac{1}{\gamma_1}\alpha^2\frac{T}{2} \\
= \;& TG(J(\delta b)) - TG(J(\delta\beta)) + \frac{1}{\gamma_1}\log(\hat{z}) + \frac{\gamma_2^{-1}}{\gamma_2^{-1} + \gamma_1^{-1}}\left[TG(J(\delta\beta)) + \gamma_1^{-1}\log(\hat{y}/\hat{z})\right] \\
& + \beta\frac{\gamma_1^{-1}}{\gamma_2^{-1} + \gamma_1^{-1}} S_0 - TG(J(\delta b)) + T\delta b J(\delta b) + \frac{1}{\gamma_1}\alpha^2\frac{T}{2} \\
= \;& \frac{\gamma_1^{-1}}{\gamma_2^{-1} + \gamma_1^{-1}}\left(T\delta b J(\delta b) - T\delta\beta J(\delta\beta)\right).
\end{aligned}$$

Since b is strictly smaller than β, the exponent in (7.29) is not zero, and the principal's utility when giving the contract $f(S_T)$ is smaller than giving the optimal contract $\hat{P}_T = \beta S_T - I_2(\hat{y} Z_T)$.

◇

Computations for Example 4.3: Consider maximizing the function

$$F(s) := U_1(d(s)) - zs = \log(d(s)) - zs \quad ,$$

for $z > 0$. Note that by L'Hospital's rule

$$\lim_{s\to\pm\infty} \frac{\log(d(s))}{s} = \lim_{s\to\pm\infty} \frac{d'(s))}{d(s)} = \lim_{s\to\pm\infty} \frac{z_d}{\hat{y}} e^{-\beta s + d(s)} \quad .$$

Thus, the limit at $s = -\infty$ is ∞ and the limit at $s = \infty$ is zero, if $C < 0$. This implies that

$$F(-\infty) = -\infty \qquad \text{and} \qquad F(\infty) = -\infty \quad .$$

Therefore, if there is a unique value $s = s(z)$ for which the first derivative of the function F is zero, then the maximum is attained at that value. This is equivalent to finding a unique value $s(z)$ such that

$$\frac{z_d}{\hat{y}} G(s(z)) - z = 0$$

where

$$G(s) = e^{-\beta s + d(s)} \quad . \quad (7.31)$$

Note that, if $C < 0$,

$$G(-\infty) = \infty \qquad \text{and} \qquad G(\infty) = 0 \ \ .$$

Therefore, it is sufficient to show that $G'(s)$ is always negative, which is equivalent to showing that

$$d'(s) < \beta \ \ . \tag{7.32}$$

In order to show this, note that

$$d'(s) = (Ei^{-1})'(x(s) + C)\beta x(s), \quad \text{where } x(s) = -\frac{z_d}{\beta \hat{y}} e^{-\beta s} < 0 \ \ .$$

It is easily verified that the maximum of $d'(s)$ over $C \leq 0$ is attained at $C = 0$. Thus, from the last equation, in order to prove (7.32), we need to show that

$$(Ei^{-1})'(x)x < 1$$

for $x < 0$. This is equivalent to

$$Ei'(Ei^{-1}(x)) < x \ \ .$$

By transforming the variables as

$$x = Ei(y)$$

the last inequality is equivalent to

$$Ei'(y) < Ei(y)$$

or

$$Ei(y) > e^y/y, \quad y < 0 \ \ .$$

Finally, by integration by parts, we have

$$Ei(y) \ = \ \int_{-\infty}^{y} \frac{e^t}{t} dt \ = \ \frac{e^y}{y} + \int_{-\infty}^{y} \frac{e^t}{t^2} dt \ > \ \frac{e^y}{y} \ \ .$$

Thus, we have shown that (7.32) holds and so there is a unique value $s(z)$ that maximizes $U_1(d(s)) - sz$.

It remains to show that $z_d > 0$ and $C < 0$ can be chosen so that

$$E[Z_T s(z_d Z_T)] = S_0 \qquad \text{and} \qquad E[U_1(d(S_T))] = R \ \ , \tag{7.33}$$

where the process S is defined by (7.23). Recall that

$$s(z) = G^{-1}_{z_d,C}(z\hat{y}/z_d)$$

where the function $G_{z_d,C}$ is defined in (7.31). Hence, we need to have

$$E[Z_T G^{-1}_{z_d,C}(\hat{y} Z_T)] = S_0 \ \ . \tag{7.34}$$

From the definition of the function $d(\cdot)$, it can be seen that the left-hand side above covers all values $S_0 \in (-\infty, \infty)$ as z_d ranges through $(0, \infty)$ for fixed $C < 0$, and similarly, as C ranges through $(-\infty, 0)$ for fixed $z_d > 0$. The second condition in (7.33) becomes

$$E\left[\log\left(d_{z_d,C}\left(G^{-1}_{z_d,C}(\hat{y}Z_T)\right)\right)\right] = R \quad .$$

Again, it can be seen that the left-hand side above covers all values $R \in (-\infty, \infty)$ as z_d ranges through $(0, \infty)$ for fixed $C < 0$, and as C ranges through $(-\infty, 0)$ for fixed $z_d > 0$. We conclude that there exist values $z_d > 0, C < 0$ so that (7.33) holds.

$\diamond$

Proof of Theorem 5.1: We present an extension of the above argument. Consider first the first-best solution, that is, the problem of the principal maximizing her utility, under the IR constraint for the principal. Recall the dual function

$$U_2^*(y, \lambda) \;:=\; \max_{s,p} \{U_2(\beta s - p) + \lambda U_1(p) - y\beta s\} \tag{7.35}$$

for $\lambda > 0$, and y in the effective domain of U_2^*. Also recall that the maximum is attained at

$$\beta\hat{s} - \hat{p} = I_2(y) \;; \quad \hat{p} = I_1(y/\lambda) \quad . \tag{7.36}$$

Set now $\beta s = \beta S_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt$ and $p = P_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt$ in (7.35). Then, we have

$$\begin{aligned} U_2^*(\hat{Z}_T, \lambda) \;&\geq\; U_2(\beta S_T - P_T) + \lambda U_1\left(P_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right) \\ &-\hat{Z}_T\left(\beta S_T - \int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right). \end{aligned}$$

Taking expectations, we then obtain

$$E\left[U_2(\beta S_T - P_T)\right] \;\leq\; E\left[U_2^*(\hat{Z}_T, \lambda)\right] + L(\hat{Z}) - \lambda R \quad , \tag{7.37}$$

where R is the reservation utility for the agent. Note that for a given λ, this is an upper bound on the principal's utility. By (7.36), the upper bound is attained if (5.8) – (5.11) are satisfied, if we set $\lambda = 1/\bar{c}$. Thus, $\hat{a}, \hat{\sigma}$ are optimal for the principal's first best problem.

Let us now take into consideration the agent's problem. Suppose that the principal offers the agent a contract (5.11). Then, by the definition of $\tilde{U}_1$ (see (5.6)), we have the following upper bound on the agent's utility, for any stochastic process $Y \in \mathcal{D}_L$:

$$\begin{aligned} E\left[U_1\left(\hat{P}_T - \textstyle\int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right)\right] &= E\left[U_1\left(\beta S_T - I_2(\hat{Z}_T) - \textstyle\int_0^T \bar{G}(t, a_t, \sigma_t, S_t)dt\right)\right] \\ &\leq E\left[\tilde{U}_1(Y_T) + L(Y) - Y_T I_2(\hat{Z}_T)\right] \quad . \end{aligned} \tag{7.38}$$

Moreover, the smallest such bound is obtained by taking the infimum of the right-hand side, and it is attained at $Y = \bar{c}\hat{Z}$, by definition of $\hat{Z}$.

This smallest upper bound will be attained if (5.8) is satisfied and if (5.9) is satisfied with $\hat{Z}$ replaced by $\bar{c}\hat{Z}$. This will indeed be satisfied if the agent uses the first-best controls $\hat{a}, \hat{\sigma}$, because $L(\bar{c}\hat{Z}) = \bar{c}L(\hat{Z})$.

$\diamond$

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